

# BRIDGELAND STABILITY CONDITIONS ON THE ACYCLIC TRIANGULAR QUIVER

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ABSTRACT. Using results in a previous paper “Non-semistable exceptional objects in hereditary categories”, we focus here on studying the topology of the space of Bridgeland stability conditions on  $D^b(\text{Rep}_k(Q))$ , where  $Q = \begin{array}{ccc} & \circ & \\ \nearrow & & \searrow \\ \circ & \xrightarrow{\quad} & \circ \end{array}$ . In particular, we prove that this space is contractible (in the previous paper it was shown that it is connected).

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## 1. INTRODUCTION

In 1994 Maxim Kontsevich interpreted a duality coming from physics in a consistent, powerful mathematical framework called Homological Mirror Symmetry (HMS). HMS is now the foundation of a wide range of contemporary mathematical research. Numerous works by many authors have demonstrated the interaction of mirror symmetry and HMS with a wide range of new and subtle mathematical structures. One of this structures is the moduli space of stability conditions.

The study of stability in triangulated categories was initiated by M. Douglas and mathematically by T. Bridgeland. The majority of the activity since then has focused on categories of algebro-geometric origin. Significant work in this direction is due to T. Bridgeland, A. King, E. Macrì, S. Okada, Y. Toda, A. Bayer, J. Woolf, J. Collins, A. Polishchuck et. al.

In previous works [9], [8, joint with F. Haiden and M. Kontsevich] we developed results and ideas by T. Bridgeland [1], A. King [13], E. Macrì [16], J. Collins and A. Polishchuck [5].

Recently in [17] J. Woolf showed classes of categories with contractible component in the space of stability conditions. His paper generalizes and unifies various known results for stability spaces of specific categories, and settles some conjectures about the stability spaces associated to Dynkin quivers, and to their Calabi-Yau-N Ginzburg algebras. However the results in [17] do not cover tame representation type quivers, these quivers are beyond the scope of [17].

In the present paper we give a new example of a tame representation type quiver with contractible space of stability conditions. This paper is a natural consequence of our previous paper [9]. Both are based on ideas of E. Macrì, which he gave in [16] studying  $\text{Stab}(D^b(K(l)))$ , where  $K(l)$  is the  $l$ -Kronecker quiver.

1.1. T. Bridgeland defined in [1] the space of stability conditions on a triangulated category  $\mathcal{T}$ , denoted by  $\text{Stab}(\mathcal{T})$ , and proved that it is a complex manifold on which act the groups  $\widetilde{GL}^+(2, \mathbb{R})$  and  $\text{Aut}(\mathcal{T})$ . To any bounded t-structure of  $\mathcal{T}$  he assigned a family of stability conditions.

E. Macrì constructed in [16] stability conditions using exceptional collections and the action of  $\widetilde{GL}^+(2, \mathbb{R})$  on  $\text{Stab}(\mathcal{T})$ . Applying results in [4], he showed that the extension closure of a full Ext-exceptional collection<sup>1</sup>  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  in  $\mathcal{T}$  is a bounded t-structure. The stability conditions obtained from this t-structure together with their translations by the right action of  $\widetilde{GL}^+(2, \mathbb{R})$  will be referred to as *generated by*  $\mathcal{E}$ .

E. Macrì, studying  $\text{Stab}(D^b(K(l)))$  in [16], gave an idea for producing an exceptional pair generating a given stability condition  $\sigma$  on  $D^b(K(l))$ , where  $K(l)$  is the  $l$ -Kronecker quiver.

We defined in [9] the notion of a  $\sigma$ -exceptional collection ([9, Definition 3.19]), so that the full  $\sigma$ -exceptional collections are exactly the exceptional collections which generate  $\sigma$ , and we focused on constructing  $\sigma$ -exceptional collections from a given  $\sigma \in \text{Stab}(D^b(\mathcal{A}))$ , where  $\mathcal{A}$  is a hereditary, hom-finite, abelian category. We developed tools for constructing  $\sigma$ -exceptional collections of length at least three in  $D^b(\mathcal{A})$ . These tools are based on the notion of *regularity-preserving hereditary category*, introduced in [9] to avoid difficulties related to the Ext-nontrivial couples (couples of exceptional objects in  $\mathcal{A}$  with  $\text{Ext}^1(X, Y) \neq 0$  and  $\text{Ext}^1(Y, X) \neq 0$ ).

After a detailed study of the exceptional objects of the affine quiver  $Q$  (see figure (1) below) it was shown in [9] that  $\text{Rep}_k(Q)$  is regularity preserving and the newly obtained methods for constructing  $\sigma$ -triples were applied to the case  $\mathcal{A} = \text{Rep}_k(Q)$ . As a result we obtained the following theorem:

**Theorem 1.1** ([9]). *Let  $k$  be an algebraically closed field. For each  $\sigma \in \text{Stab}(D^b(\text{Rep}_k(Q)))$  there exists a full  $\sigma$ -exceptional collection.*

In other words, all stability conditions on  $D^b(Q)$  are generated by exceptional collections (in this case exceptional triples). This theorem implies that  $\text{Stab}(D^b(Q))$  is connected [9, Corollary 10.2].

Using Theorem 1.1 and the data about the exceptional collections given in [9, Section 2], we prove here the following:

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<sup>1</sup>An exceptional collection  $\mathcal{E}$  is said to be *Ext-exceptional* if  $\text{Hom}^{\leq 0}(E_i, E_j) = 0$  for  $0 \leq i < j \leq n$ .

**Theorem 1.2.** *Let  $k$  be an algebraically closed field. Let  $Q$  be the following quiver:*

$$(1) \quad Q = \begin{array}{ccc} & \circ & \\ \nearrow & & \nwarrow \\ \circ & \longrightarrow & \circ \end{array}$$

*The space of Bridgeland stability conditions  $\text{Stab}(D^b(\text{Rep}_k(Q)))$  is a contractible (and connected) manifold, where  $D^b(\text{Rep}_k(Q))$  is the derived category of representations of  $Q$ .*

1.2. We give now more details about the structure of  $\text{Stab}(D^b(\text{Rep}_k(Q)))$  and about the proof of Theorem 1.2.

We call an exceptional pair  $(E, F)$  in  $D^b(\text{Rep}_k(Q))$  a 2-Kronecker pair if  $\text{hom}^{\leq 0}(E, F) = 0$ , and  $\text{hom}^1(E, F) = 2$ . Recall that the Braid group on two strings  $B_2 \cong \mathbb{Z}$  acts on the set of equivalence classes of exceptional pairs in  $\mathcal{T}$ .<sup>2</sup> The set of equivalence classes of 2-Kronecker pairs is invariant under the action of  $B_2$ . In Subsection 3.1 are described the orbits of this action on the 2-Kronecker pairs (using [9, Corollary 2.9]). There are two such orbits and in terms of our notations they are  $\{(a^m, a^{m+1}[-1])\}_{m \in \mathbb{Z}}$  and  $\{(b^m, b^{m+1}[-1])\}_{m \in \mathbb{Z}}$  (see Remark 3.14).

It turns out that the exceptional objects of  $D^b(\text{Rep}_k(Q))$  can be grouped as follows  $\{a^m\}_{m \in \mathbb{Z}} \cup \{M, M'\} \cup \{b^m\}_{m \in \mathbb{Z}}$ , where  $\{M, M'\} \subset \text{Rep}_k(Q)$  is the unique Ext-nontrivial couple of  $D^b(\text{Rep}_k(Q))$ .

Let  $\mathfrak{T}_a^{st}$  and  $\mathfrak{T}_b^{st}$  be the stability conditions generated by the exceptional triples containing a subsequence of the from  $(a^m[p], a^{m+1}[q])$  and  $(b^m[p], b^{m+1}[q])$  for some  $m, p, q \in \mathbb{Z}$ , respectively. Using Theorem 1.1 we show in Section 4 that  $\text{Stab}(D^b(\text{Rep}_k(Q))) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}$ , where  $(\_, M, \_) \cup (\_, M', \_)$  denotes the set of stability conditions generated by triples of the form  $(A, M[p], C)$  or  $(A, M'[p], C)$  with  $p \in \mathbb{Z}$  (these turn out to be the triples  $(A, B, C)$  for which  $\dim(\text{Hom}^i(A, B)) \leq 1$ ,  $\dim(\text{Hom}^i(A, C)) \leq 1$ ,  $\dim(\text{Hom}^i(B, C)) \leq 1$  for all  $i \in \mathbb{Z}$ ).

The main steps are as follows. In Section 5 we show that  $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$ . In Section 6 we show that  $\mathfrak{T}_a^{st}$  and  $\mathfrak{T}_b^{st}$  are contractible. In Section 7 we connect  $\mathfrak{T}_a^{st}$  and  $\mathfrak{T}_b^{st}$  by  $(\_, M, \_) \cup (\_, M', \_)$  and show that in this procedure the contractibility is preserved.

The theorem from topology which we use to glue stability conditions generated by different exceptional triples is the Seifert-van Kampen theorem, modified about contractile subsets in manifolds (see Remark A.6). In Section 2 are given several important tools, which we use throughout to analyze the intersection of the sets of stability conditions generated by different exceptional collections. These tools are extensions of results and ideas in [13], [16], [8], [9]. In the final step (Section 7) we utilize as such a tool also the relation  $R \dashrightarrow (S, E)$  between a  $\sigma$ -regular object  $R$  and an exceptional pair generated by it (introduced in [9]).

In Section 3 we organize in a better way the obtained in [9, Section 2] data about  $\text{Hom}(X, Y)$  and  $\text{Ext}^1(X, Y)$ , where  $X, Y$  vary throughout the exceptional objects of  $\text{Rep}_k(Q)$ , and we add some observations about the behavior of the central charges of the exceptional objects, which are very essential for the proof of Theorem 1.2 as well.

Today, in view of the parallel between dynamical systems and categories [8], [3] and in view of the Motivic Donaldson Thomas invariants [12] the importance of studying the topology of the space of Bridgeland stability conditions is even bigger.

We still do not understand the meaning of the obtained picture about  $\text{Stab}(D^b(\text{Rep}_k(Q)))$ . We hope that an understanding of this meaning will open a way to analyzing more cases.

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<sup>2</sup>Here we take the equivalence  $\sim$  explained in **Some notations** and it is clear when a given equivalence class w.r.  $\sim$  will be called a 2-Kronecker pair

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**Some notations.** In these notes the letters  $\mathcal{T}$  and  $\mathcal{A}$  denote always a triangulated category and an abelian category, respectively, linear over an algebraically closed field  $k$ .  $T$

the shift functor in  $\mathcal{T}$  is designated by  $[1]$ . We write  $\text{Hom}^i(X, Y)$  for  $\text{Hom}(X, Y[i])$  and  $\text{hom}^i(X, Y)$  for  $\dim_k(\text{Hom}(X, Y[i]))$ , where  $X, Y \in \mathcal{T}$ . For  $X, Y \in \mathcal{A}$ , writing  $\text{Hom}^i(X, Y)$ , we consider  $X, Y$  as elements in  $\mathcal{T} = D^b(\mathcal{A})$ , i.e.  $\text{Hom}^i(X, Y) = \text{Ext}^i(X, Y)$ .

We write  $\langle S \rangle \subset \mathcal{T}$  for the triangulated subcategory of  $\mathcal{T}$  generated by  $S$ , when  $S \subset \text{Ob}(\mathcal{T})$ .

An *exceptional object* is an object  $E \in \mathcal{T}$  satisfying  $\text{Hom}^i(E, E) = 0$  for  $i \neq 0$  and  $\text{Hom}(E, E) = k$ . We denote by  $\mathcal{A}_{exc}$ , resp.  $D^b(\mathcal{A})_{exc}$ , the set of all exceptional objects of  $\mathcal{A}$ , resp. of  $D^b(\mathcal{A})$ .

An *exceptional collection* is a sequence  $\mathcal{E} = (E_0, E_1, \dots, E_n) \subset \mathcal{T}_{exc}$  satisfying  $\text{hom}^*(E_i, E_j) = 0$  for  $i > j$ . If in addition we have  $\langle \mathcal{E} \rangle = \mathcal{T}$ , then  $\mathcal{E}$  will be called a full exceptional collection. For a vector  $\mathbf{p} = (p_0, p_1, \dots, p_n) \in \mathbb{Z}^{n+1}$  we denote  $\mathcal{E}[\mathbf{p}] = (E_0[p_0], E_1[p_1], \dots, E_n[p_n])$ . Obviously  $\mathcal{E}[\mathbf{p}]$  is also an exceptional collection. The exceptional collections of the form  $\{\mathcal{E}[\mathbf{p}] : \mathbf{p} \in \mathbb{Z}^{n+1}\}$  will be said to be shifts of  $\mathcal{E}$ .

For two exceptional collections  $\mathcal{E}_1, \mathcal{E}_2$  of equal length we write  $\mathcal{E}_1 \sim \mathcal{E}_2$  if  $\mathcal{E}_2 \cong \mathcal{E}_1[\mathbf{p}]$  for some  $\mathbf{p} \in \mathbb{Z}^{n+1}$ .

An abelian category  $\mathcal{A}$  is said to be hereditary, if  $\text{Ext}^i(X, Y) = 0$  for any  $X, Y \in \mathcal{A}$  and  $i \geq 2$ , it is said to be of finite length, if it is Artinian and Noetherian.

For any quiver  $Q$  we denote by  $D^b(\text{Rep}_k(Q))$  or just by  $D^b(Q)$  the derived category of the category of representations of  $Q$ .

For any  $a \in \mathbb{R}$  and any complex number  $z \in e^{i\pi a} \cdot (\mathbb{R} + i\mathbb{R}_{>0})$ , respectively  $z \in e^{i\pi a} \cdot (\mathbb{R}_{<0} \cup (\mathbb{R} + i\mathbb{R}_{>0}))$ , we denote by  $\arg_{(a, a+1)}(z)$ , resp.  $\arg_{(a, a+1]}(z)$ , the unique  $\phi \in (a, a+1)$ , resp.  $\phi \in (a, a+1]$ , satisfying  $z = |z| \exp(i\pi\phi)$ .

For a non-zero complex number  $v \in \mathbb{C}$  we denote the two connected components of  $\mathbb{C} \setminus \mathbb{R}v$  by:

$$(2) \quad v_+^c = v \cdot (\mathbb{R} + i\mathbb{R}_{>0}) \quad v_-^c = v \cdot (\mathbb{R} - i\mathbb{R}_{>0}) \quad v \in \mathbb{C} \setminus \{0\}.$$

For  $b \in (a, a+1)$ ,  $c \in (a-1, a)$   $r_1 > 0$ ,  $r_2 > 0$  we have

$$(3) \quad \begin{aligned} \arg_{(a, a+1)}(r_1 \exp(i\pi a) + r_2 \exp(i\pi b)) &= a + \arg_{(0, 1)}(r_1 + r_2 \exp(i\pi(b-a))) \\ \arg_{(a-1, a)}(r_1 \exp(i\pi a) + r_2 \exp(i\pi c)) &= a + \arg_{(-1, 0)}(r_1 + r_2 \exp(i\pi(c-a))). \end{aligned}$$

These formulas imply that for  $c \in (a-1, a)$ ,  $r_1 > 0$ ,  $r_2 > 0$  we have

$$(4) \quad \arg_{(a-1, a)}(r_1 \exp(i\pi a) + r_2 \exp(i\pi c)) = -\arg_{(-a, -a+1)}(r_1 \exp(-i\pi a) + r_2 \exp(-i\pi c)).$$

## 2. SOME GENERAL REMARKS

Here we give tools which will be used throughout to analyze the intersection of the sets of stability conditions generated by different exceptional collections (Propositions 2.2, 2.9, 2.10 and

Lemmas 2.11, 2.12). A description of the set of stability conditions generated by all shifts of a fixed exceptional triple is given in Proposition 2.7, which is also important for the rest of the paper. Due to Remark 2.5 it seems that Proposition 2.7 can not be generalized straightforwardly to the case of exceptional collections of length bigger than 3.

**2.1. Basic facts and notations related to Bridgeland stability conditions.** We use freely the axioms and notations on stability conditions introduced by Bridgeland in [1] and some additional notations used in [9, Subsection 3.2]. In particular, for  $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$  we denote by  $\sigma^{ss}$  the set of  $\sigma$ -semistable objects, i. e.

$$(5) \quad \sigma^{ss} = \cup_{t \in \mathbb{R}} \mathcal{P}(t) \setminus \{0\}.$$

For any interval  $I \subset \mathbb{R}$  the extension closure of the slices  $\{\mathcal{P}(x)\}_{x \in I}$  is denoted by  $\mathcal{P}(I)$  in [1]. The nonzero objects in the subcategory  $\mathcal{P}(I)$  are exactly those  $X \in \mathcal{T} \setminus \{0\}$ , which satisfy  $\phi_{\pm}(X) \in I$ , i. e. whose HN factors have phases in  $I$ . In particular, if  $X \in \mathcal{P}(a-1, a] \setminus \{0\}$  then  $Z(X) \in \exp(i\pi a)_{>0}^{\subset} \cup \mathbb{R}_{>0} \exp(i\pi a)$ .

From [1] we know that for any  $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$  and any  $t \in \mathbb{R}$  the subcategory  $\mathcal{P}(t, t+1]$  is a heart of a bounded t-structure. In particular  $\mathcal{P}(t, t+1]$  is an abelian category, whose short exact sequences are exactly these sequences  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  with  $A, B, C \in \mathcal{P}(t, t+1]$ , s. t. for some  $\gamma : C \rightarrow A[1]$  the sequence  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$  is a triangle in  $\mathcal{T}$ . Using these remarks, the HH filtration and by drawing pictures one easily shows the following properties:

**Remark 2.1.** *Let  $t \in \mathbb{R}$  and  $X \in \mathcal{P}(a-1, a]$ . Then:*

- (a) *If  $X \notin \sigma^{ss}$  then  $\phi_{-}(X) < \arg_{(a-1, a]}(Z(X)) < \phi_{+}(X)$ .*
- (b)  *$X \notin \sigma^{ss}$  iff there exists a monic arrow  $X' \rightarrow X$  in the abelian category  $\mathcal{P}(a-1, a]$  satisfying  $\arg_{(a-1, a]}(Z(X')) > \arg_{(a-1, a]}(Z(X))$ .*
- (c) *If  $Z(X) \in v_{+}^{\subset}$  for some  $v \in \mathbb{C}^{*}$  with  $v = |v| \exp(i\pi t)$  and  $a-1 \in (t, t+1)$  or  $a \in (t, t+1)$ , then  $\arg_{(a-1, a]}(Z(X)) = \arg_{(t, t+1)}(Z(X))$ . In particular, when  $X \in \sigma^{ss}$ , we have:  $\phi(X) = \arg_{(t, t+1)}(Z(X))$ .*

**2.2. Some remarks on  $\sigma$ -exceptional collections.** E. Macrì proved in [16, Lemma 3.14] that the extension closure  $\mathcal{A}_{\mathcal{E}}$  of a full Ext-exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  in  $\mathcal{T}$  is a heart of a bounded t-structure. Furthermore,  $\mathcal{A}_{\mathcal{E}}$  is of finite length and  $E_0, E_1, \dots, E_n$  are the simple objects in it. By Bridgeland's [1, Proposition 5.3] from the bounded t-structure  $\mathcal{A}_{\mathcal{E}}$  is produced a family of stability conditions, which we denote by  $\mathbb{H}^{\mathcal{A}_{\mathcal{E}}} \subset \text{Stab}(\mathcal{T})$  or sometimes just  $\mathbb{H}^{\mathcal{E}} \subset \text{Stab}(\mathcal{T})$ .

For a given  $\sigma \in \text{Stab}(\mathcal{T})$  we define a  $\sigma$ -exceptional collection ([9, Definition 3.19]) as an Ext-exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$ , s. t. the objects  $\{E_i\}_{i=0}^n$  are  $\sigma$ -semistable, and  $\{\phi(E_i)\}_{i=0}^n \subset (t, t+1)$  for some  $t \in \mathbb{R}$ . The following Proposition is basic for this paper:

**Proposition 2.2.** *Let  $\mathcal{T}$  be a  $k$ -linear triangulated category and  $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$ . Let  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  be a full  $\sigma$ -exceptional collection such that  $\phi(E_i) \geq \phi(E_{i+1})$  and  $\text{hom}^1(E_i, E_{i+1}) \neq 0$  for some  $i \in \{0, 1, \dots, n-1\}$ . Let  $\mathcal{A}_{i, i+1}$  be the extension closure of  $E_i, E_{i+1}$  in  $\mathcal{T}$ . Then each element in  $\mathcal{T}_{exc} \cap \mathcal{A}_{i, i+1}$  is semistable.*

*Proof.* If  $\phi(E_i) = \phi(E_{i+1}) = t$ , then  $\mathcal{A}_{i, i+1} \subset \mathcal{P}(t)$  and hence all non-zero objects in  $\mathcal{A}_{i, i+1}$  are semistable, therefore we can assume that  $\phi(E_i) > \phi(E_{i+1})$ .

By [9, Corollary 3.20] we have  $\sigma \in \Theta'_\mathcal{E} = \mathbb{H}^\mathcal{E} \cdot \widetilde{GL}^+(2, \mathbb{R})$ . Since the action of  $\widetilde{GL}^+(2, \mathbb{R})$  does not change the order of the phases, we can assume that  $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^\mathcal{E}$ , which means that the extension closure of  $\mathcal{E}$  is the t-structure  $\mathcal{P}(0, 1]$  and

$$(6) \quad \phi(E_j) = \arg_{(0,1]}(Z(E_j)) \quad j = 1, \dots, n.$$

Let us denote  $\mathcal{T}_{i,i+1} = \langle E_i, E_{i+1} \rangle$ . From [9, Proposition 3.17] we have a projection map  $\mathbb{H}^\mathcal{E} \rightarrow \mathbb{H}^{\mathcal{A}_{i,i+1}} \subset \text{Stab}(\mathcal{T}_{i,i+1})$  and it maps  $\sigma = (\mathcal{P}, Z)$  to a stability condition  $\sigma' = (\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{A}_{i,i+1}}$  with  $Z'(E_i) = Z(E_i)$ ,  $Z'(E_{i+1}) = Z(E_{i+1})$  and  $\{\mathcal{P}'(t) = \mathcal{P}(t) \cap \mathcal{T}_{i,i+1}\}_{t \in \mathbb{R}}$ . Therefore it remains to show that the objects in  $\mathcal{T}_{exc} \cap \mathcal{A}_{i,i+1}$  are  $\sigma'$ -semistable.

From [8, Lemma 3.22] we have that  $\mathcal{A}_{i,i+1}$  is a bounded t-structure in  $\mathcal{T}_{i,i+1}$  and an equivalence of abelian categories  $F : \mathcal{A}_{i,i+1} \rightarrow \text{Rep}_k(K(l))$  with  $F(E_i) = s_1$ ,  $F(E_{i+1}) = s_2$ , where  $l = \text{hom}^1(E_i, E_{i+1})$  and  $s_1, s_2$  are the simple representations of  $K(l)$  with  $k$  at the source, sink, respectively. This equivalence maps  $\sigma' \in \mathbb{H}^{\mathcal{A}_{i,i+1}}$  to a stability condition

$$\sigma'' = (\mathcal{P}'', Z'') \in \mathbb{H}^{\text{Rep}_k(K(l))} \subset \text{Stab}(D^b(K(l))) \quad Z(E_i) = Z''(s_1), Z(E_{i+1}) = Z''(s_2).$$

If  $E \in \mathcal{T}_{exc} \cap \mathcal{A}_{i,i+1}$ , then by the fact that  $F$  is an equivalence of abelian categories it follows that  $F(E) \in \text{Rep}_k(K(l))$  is an exceptional representation. Since  $\{F(\mathcal{P}'(t)) = \mathcal{P}''(t)\}_{t \in (0,1]}$ , it remains to show that each exceptional representation of  $\text{Rep}_k(K(l))$  is  $\sigma''$ -semistable.

Let  $\rho \in \text{Rep}_k(K(l))_{exc}$ . Then the dimension vector  $\underline{\dim}(\rho) = (n, m) \in (n, m)$  is a real root of  $K(l)$ , furthermore it is a Schur root. From (6) we have  $\arg(Z''(s_1)) > \arg(Z''(s_2))$ . By the arguments in the proof of [8, Lemma 3.19] using a theorem by King ([13, Proposition 4.4]) and  $\arg(Z''(s_1)) > \arg(Z''(s_2))$  we obtain a  $\sigma''$ -stable representation  $X \in \text{Rep}_k(K(l))$  with  $\underline{\dim}(X) = (n, m)$ . Since  $X$  is stable, it is simple in  $\mathcal{P}''(t)$ , where  $t = \phi''(X)$ , in particular it is indecomposable in  $\mathcal{P}''(t)$ . Since  $\mathcal{P}''(t)$  is a thick subcategory (see [9, Lemma 3.7]), it follows that  $X$  is indecomposable in  $\text{Rep}_k(K(l))$ . Since  $\underline{\dim}(\rho)$  is a real root and both  $X, \rho$  are indecomposable representations, the equality  $\underline{\dim}(\rho) = \underline{\dim}(X)$  implies  $\rho \cong X$  (see [11, Theorem 2, c)). The proposition follows.  $\square$

Other statements, which will be widely used in the next sections are Propositions 2.7, 2.9 and 2.10. For the proof of Proposition 2.7 it is useful to define:

**Definition 2.3.** Let  $n \geq 1$  be an integer. Let  $\mathcal{I} = \{I_{ij} = (l_{ij}, r_{ij}) \subset \mathbb{R}\}_{0 \leq i < j \leq n}$  be a family of non-empty open intervals, and let  $\mathfrak{l} = \{l_{ij} \in \{-\infty\} \cup \mathbb{R}\}_{0 \leq i < j \leq n}$ ,  $\mathfrak{r} = \{r_{ij} \in \mathbb{R} \cup \{+\infty\}\}_{0 \leq i < j \leq n}$  be the corresponding families of left and right endpoints.

We will denote the following open convex set  $\{(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : y_i - y_j \in I_{ij} \text{ } i < j\} \subset \mathbb{R}^{n+1}$  by  $S^n(\mathcal{I})$  or  $S^n(\mathfrak{l}, \mathfrak{r})$ .

For a full Ext-exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  in  $\mathcal{T}$  we denote  $\Theta'_\mathcal{E} = \mathbb{H}^\mathcal{E} \cdot \widetilde{GL}^+(2, \mathbb{R})$ . If  $\mathcal{E}$  is a full Ext-exceptional collection, then we have (see [9, Remark 3.21]):

$$(7) \quad \Theta'_\mathcal{E} = \mathbb{H}^\mathcal{E} \cdot \widetilde{GL}^+(2, \mathbb{R}) = \{\sigma : \mathcal{E} \subset \sigma^{ss} \text{ and } |\phi^\sigma(E_i) - \phi^\sigma(E_j)| < 1 \text{ for } i < j\}$$

and the assignment:

$$(8) \quad \{\sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss}\} \ni (\mathcal{P}, Z) \xrightarrow{f_\mathcal{E}} (\{|Z(E_i)|\}_{i=0}^n, \{\phi^\sigma(E_i)\}_{i=0}^n) \in \mathbb{R}^{2(n+1)}$$

restricted to  $\Theta'_\mathcal{E}$  defines a homeomorphism between  $\Theta'_\mathcal{E}$  and  $\mathbb{R}_{>0}^{n+1} \times S^n(-1, +1)$  (as defined in Definition 2.3).

Assume now that  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  is any full exceptional collection in  $\mathcal{T}$  (not restricted to be Ext). If  $\mathcal{T}$  is a triangulated category of finite type, then there are infinitely many choices of  $\mathbf{p} \in \mathbb{Z}^{n+1}$  such that  $\mathcal{E}[\mathbf{p}] = (E_0[p_0], E_1[p_1], \dots, E_n[p_n])$  is an Ext-exceptional collection. [16, Lemma 3.19] says that the following open subset of stability conditions is connected and simply connected:

$$(9) \quad \Theta_{\mathcal{E}} = \bigcup_{\{\mathbf{p} \in \mathbb{Z}^{n+1} : \mathcal{E}[\mathbf{p}] \text{ is Ext}\}} \Theta'_{\mathcal{E}[\mathbf{p}]} \subset \text{Stab}(\mathcal{T}).$$

For the sake of completeness we will comment on this set as well (compare with [16, proof of Lemma 3.19]).

By [9, Corollary 3.20]  $\Theta_{\mathcal{E}}$  is the set of stability conditions  $\sigma \in \text{Stab}(\mathcal{T})$  for which a shift of  $\mathcal{E}$  is a  $\sigma$ -exceptional collection, in particular for each  $\sigma \in \Theta_{\mathcal{E}}$  we have  $\mathcal{E} \subset \sigma^{ss}$ . Hence the assignment (8) is well defined on  $\Theta_{\mathcal{E}}$ . Furthermore, this defines a homeomorphism between  $\Theta_{\mathcal{E}}$  and  $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$ . Indeed, if  $\mathcal{E}[\mathbf{p}]$  is an Ext-collection for some  $\mathbf{p} \in \mathbb{Z}^{n+1}$ , then  $f_{\mathcal{E}[\mathbf{p}]}$  maps  $\Theta'_{\mathcal{E}[\mathbf{p}]}$  homeomorphically to  $\mathbb{R}_{>0}^{n+1} \times S^n(-\mathbf{1}, +\mathbf{1})$  (see after (8)) and due to  $f_{\mathcal{E}[\mathbf{p}]} - (0, \mathbf{p}) = f_{\mathcal{E}}$  we see that  $f_{\mathcal{E}[\mathbf{p}]}$  is homeomorphism onto its image  $\mathbb{R}_{>0}^{n+1} \times (S^n(-\mathbf{1}, +\mathbf{1}) - \mathbf{p})$ . Therefore, provided that  $f_{\mathcal{E}}$  is injective on  $\Theta_{\mathcal{E}}$ , **the following restriction is a homeomorphism:**

$$(10) \quad f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \rightarrow \mathbb{R}_{>0}^{n+1} \times \left( \bigcup_{\mathbf{p} \in A} S^n(-\mathbf{1}, +\mathbf{1}) - \mathbf{p} \right), \text{ where } A = \{\mathbf{p} \in \mathbb{Z}^{n+1} : \mathcal{E}[\mathbf{p}] \text{ is Ext}\}.$$

To show that the obtained function is injective, assume that  $\sigma_i = (\mathcal{P}_i, Z_i) \in \Theta_{\mathcal{E}}$ ,  $i = 1, 2$  and  $f_{\mathcal{E}}(\sigma_1) = f_{\mathcal{E}}(\sigma_2)$ , i. e.  $|Z_1(E_j)| = |Z_2(E_j)|$ ,  $\phi^{\sigma_1}(E_j) = \phi^{\sigma_2}(E_j)$  for all  $j$ , then by (7) and the axiom  $\phi^{\sigma}(E_j[p_j]) = \phi^{\sigma}(E_j) + p_j$  we see that for any  $\mathbf{p}$  the incidence  $\sigma_1 \in \Theta'_{\mathcal{E}[\mathbf{p}]}$  is equivalent to  $\sigma_2 \in \Theta'_{\mathcal{E}[\mathbf{p}]}$ , hence by the injectivity of  $f_{\mathcal{E}[\mathbf{p}]}$  and  $f_{\mathcal{E}[\mathbf{p}]} - (0, \mathbf{p}) = f_{\mathcal{E}}$  we obtain  $\sigma_1 = \sigma_2$ . Thus, we see that (10) is a homeomorphism.

Finally, note that by (7), (8), (9), and  $f_{\mathcal{E}[\mathbf{p}]} = f_{\mathcal{E}} + (0, \mathbf{p})$  one easily shows that

$$(11) \quad \Theta_{\mathcal{E}} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \phi^{\sigma}(\mathcal{E}) \in \bigcup_{\mathbf{p} \in A} S^n(-\mathbf{1}, +\mathbf{1}) - \mathbf{p} \right\}.$$

**2.3. The set  $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$  when  $n = 2$ .** Here is given an explicit representation of  $f_{\mathcal{E}}(\Theta_{\mathcal{E}})$ , when  $n = 2$ . Remark 2.5 shows that the case  $n \geq 3$  is not completely analogous. The only statement of this subsection, which will be used later is Proposition 2.7, the rest is its proof.

Let us denote first:

$$(12) \quad B^n = \{(0, q_1, q_2, \dots, q_n) \in \mathbb{N}^{n+1} : 0 \leq q_1 \leq q_2 \leq \dots \leq q_n\}$$

The following properties are clear from the definitions of  $S^n(\mathcal{J})$  (Definition 2.3) and of  $A \subset \mathbb{Z}^{n+1}$  (formula (10))

$$(13) \quad \forall \mathbf{v} \in \text{diag}(\mathbb{R}^{n+1}) \quad S^n(\mathcal{J}) - \mathbf{v} = S^n(\mathcal{J})$$

$$(14) \quad \forall \mathbf{v} \in \text{diag}(\mathbb{Z}^{n+1}) \quad A - \mathbf{v} = A$$

$$(15) \quad \forall \mathbf{v} \in B^n \quad A - \mathbf{v} \subset A.$$

Any  $\mathbf{p} = (p_0, p_1, \dots, p_n) \in A$  can be represented as  $\mathbf{p} - (p_0, p_0, \dots, p_0) + (p_0, p_0, \dots, p_0)$ , hence if we denote

$$(16) \quad A_0 = \{\mathbf{p} \in \mathbb{Z}^{n+1} : p_0 = 0, \mathcal{E}[\mathbf{p}] \text{ is Ext}\}$$

by the properties above we can write

$$(17) \quad \bigcup_{\mathbf{p} \in A} S^n(-\mathbf{1}, +\mathbf{1}) - \mathbf{p} = \bigcup_{\mathbf{p} \in A_0} S^n(-\mathbf{1}, +\mathbf{1}) - \mathbf{p} = \bigcup_{\mathbf{p} \in A_0} \left( \bigcup_{\mathbf{v} \in B^n} S^n(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} \right) - \mathbf{p}.$$

For the cases  $n = 1, 2$  we have the following simple form of the expression in the brackets:

**Lemma 2.4.** *The following equalities hold:*

$$(18) \quad \bigcup_{\mathbf{v} \in B^1} S^1(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} = S^1(-\infty, \mathbf{1}) \quad \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} = S^2(-\infty, \mathbf{1}).$$

Recall that  $S^n(-\infty, \mathbf{1}) = \{(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : y_i - y_j < 1, i < j\}$  (see Definition 2.3).

**Remark 2.5.** *For  $n \geq 3$  we have not such an equality. For example, we have  $(0, -\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \in S^n(-\infty, \mathbf{1})$  but  $(0, -\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \notin \bigcup_{\mathbf{v} \in B^n} S^n(-\mathbf{1}, +\mathbf{1}) + \mathbf{v}$  for  $n \geq 3$ .*

*More precisely, it holds  $\bigcup_{\mathbf{v} \in B^n} S^n(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} \subsetneq S^n(-\infty, \mathbf{1})$  for  $n \geq 3$ .*

*Proof.* (of Lemma 2.4) Note first that for any  $\mathcal{J} = \{I_{ij} : i < j\}$  as in Definition 2.3 and any  $\mathbf{p} \in \mathbb{Z}^{n+1}$  we have

$$(19) \quad S^n(\{I_{ij} : i < j\}) - \mathbf{p} = S^n(\{I_{ij} - (p_i - p_j) : i < j\}).$$

In particular for  $n = 1$  we have (now the index set of  $\mathcal{J}$  has only one element:  $(0, 1)$ ):

$$\begin{aligned} \bigcup_{\mathbf{v} \in B^1} S^1(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} &= \bigcup_{(0, k) \in \mathbb{N}^2} S^1(-\mathbf{1}, +\mathbf{1}) + (0, k) = \bigcup_{k \in \mathbb{N}} S^1(-1 - k, 1 - k) \\ &= \bigcup_{k \in \mathbb{N}} \{-1 - k < y_0 - y_1 < 1 - k\} = \{y_0 - y_1 < 1\} = S^1(-\infty, +\mathbf{1}) \end{aligned}$$

Using (13) and (19) one easily shows that:

$$\begin{aligned} \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} &= \text{diag}(\mathbb{R}^{n+1}) \oplus \{y_2 = 0\} \cap \left( \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} \right) \\ S^2(-\infty, \mathbf{1}) &= \text{diag}(\mathbb{R}^{n+1}) \oplus \{y_2 = 0\} \cap S^2(-\infty, \mathbf{1}). \end{aligned}$$

Obviously we have

$$\{y_2 = 0\} \cap S^2(-\infty, \mathbf{1}) = \{y_2 = 0\} \cap \left\{ \begin{array}{l} y_0 - y_1 < 1 \\ y_0 - y_2 < 1 \\ y_1 - y_2 < 1 \end{array} \right\} = \left\{ \begin{array}{l} y_0 - y_1 < 1 \\ y_0 < 1 \\ y_1 < 1 \end{array} \right\}.$$

We will prove the second equality in (18) by showing that:

$$(20) \quad \{y_2 = 0\} \cap \left( \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, +\mathbf{1}) + \mathbf{v} \right) = \left\{ \begin{array}{l} y_0 - y_1 < 1 \\ y_0 < 1 \\ y_1 < 1 \end{array} \right\}.$$



Let  $(0, k, k+l) \in B^2$ ,  $k, l \in \mathbb{N}$  be a vector in  $B^2$ . By (19) we have:

$$(21) \quad S^2(-\mathbf{1}, \mathbf{1}) + (0, k, k+l) = \left\{ \begin{array}{c} -1-k < y_0 - y_1 < 1-k \\ -1-k-l < y_0 - y_2 < 1-k-l \\ -1-l < y_1 - y_2 < 1-l \end{array} \right\} \subset \left\{ \begin{array}{c} y_0 - y_1 < 1 \\ y_0 - y_2 < 1 \\ y_1 - y_2 < 1 \end{array} \right\}.$$

Denoting the unit open square by  $C(-1, +1) = \{|y_i| < 1; i = 0, 1\} \subset \mathbb{R}^2$ , we can write:

$$\begin{aligned} \{y_2 = 0\} \cap (S^2(-\mathbf{1}, \mathbf{1}) + (0, k, k+l)) &= \left\{ \begin{array}{c} -1-k < y_0 - y_1 < 1-k \\ -1-k-l < y_0 < 1-k-l \\ -1-l < y_1 < 1-l \end{array} \right\} \\ &= S^1(-1-k, +1-k) \cap (C(-1, +1) - (k+l, l)) \\ &= (S^1(-1, +1) + (0, k)) \cap (C(-1, +1) - (k+l, l)) \\ &= (S^1(-1, +1) - (k+l, k+l) + (0, k)) \cap (C(-1, +1) - (k+l, l)) \\ &= (S^1(-1, +1) \cap C(-1, +1)) - (k+l, l). \end{aligned}$$

Therefore:

$$(22) \quad \{y_2 = 0\} \cap \left( \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, \mathbf{1}) + \mathbf{v} \right) = \bigcup_{k \in \mathbb{N}} \left( \bigcup_{l \in \mathbb{N}} (S^1(-1, 1) \cap C(-1, 1)) - (l, l) \right) - (k, 0).$$

Before we continue with the proof of Lemma 2.4, we prove:

**Lemma 2.6.** *For any  $k \in \mathbb{Z} \cup \{+\infty\}$  we have the following equality:*

$$(23) \quad \bigcup_{l \leq k} S^1(-1, +1) \cap C(-1, +1) + (l, l) = S^1(-1, +1) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\}.$$

*Proof.* We show first the equality for  $k = +\infty$ . Let  $(a_0, a_1) \in S^1(-1, +1)$ , i. e.  $|a_0 - a_1| < 1$ . Since  $\mathbb{R} = \bigcup_{l \in \mathbb{Z}} [2l-1, 2l+1)$ , there exists  $l \in \mathbb{Z}$  such that  $a_0 + a_1 \in [2l-1, 2l+1)$ , i. e.  $-1 \leq a_0 + a_1 - 2l \leq 1$ . We have also  $-1 < a_0 - a_1 < +1$  and due to the equalities:

$$a_0 - l = \frac{a_0 + a_1 - 2l}{2} + \frac{a_0 - a_1}{2}; \quad a_1 - l = \frac{a_0 + a_1 - 2l}{2} + \frac{a_1 - a_0}{2}$$

we obtain  $-1 = -\frac{1}{2} - \frac{1}{2} < a_i - l < \frac{1}{2} + \frac{1}{2} = 1$  for  $i = 0, 1$ . Hence  $(a_0, a_1) - (l, l) \in C(-1, +1) \cap S(-1, +1)$ , and we proved the equality (23) with  $k = +\infty$ . By (13) and since the translation in  $\mathbb{R}^2$  is bijective we rewrite this equality as follows  $S^1(-1, +1) = \bigcup_{l \in \mathbb{Z}} S^1(-1, +1) \cap C(-1, +1) + (l, l) = \bigcup (S^1(-1, +1) + (l, l)) \cap (C(-1, +1) + (l, l)) = S^1(-1, +1) \cap (\bigcup_{l \in \mathbb{Z}} C(-1, +1) + (l, l))$ . Hence

$$(24) \quad S^1(-1, 1) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\} = S^1(-1, 1) \cap \left( \bigcup_{l \in \mathbb{Z}} C(-1, 1) + (l, l) \right) \cap \left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\}.$$

Due to the equalities

$$\left\{ \begin{array}{c} y_0 < 1+k \\ y_1 < 1+k \end{array} \right\} \cap (C(-1, 1) + (l, l)) = \begin{cases} \emptyset & \text{if } l \geq k+2 \\ \left\{ \begin{array}{c} k < y_0 < 1+k \\ k < y_1 < 1+k \end{array} \right\} \subset C(-1, 1) + (k, k) & \text{if } l = k+1 \\ C(-1, +1) + (l, l) & \text{if } l \leq k \end{cases}$$

we obtain  $(\bigcup_{l \in \mathbb{Z}} C(-1, 1) + (l, l)) \cap \left\{ \begin{array}{l} y_0 < 1 + k \\ y_1 < 1 + k \end{array} \right\} = \bigcup_{l \leq k} C(-1, 1) + (l, l)$ . By (24) and applying again (13) we obtain the equality (23) for  $k \in \mathbb{Z}$ .  $\square$

Now we put (23) with  $k = 0$  in (22) and obtain

$$(25) \quad \{y_2 = 0\} \cap \left( \bigcup_{\mathbf{v} \in B^2} S^2(-\mathbf{1}, \mathbf{1}) + \mathbf{v} \right) = \bigcup_{k \in \mathbb{N}} \left( S^1(-1, +1) \cap \left\{ \begin{array}{l} y_0 < 1 \\ y_1 < 1 \end{array} \right\} \right) - (k, 0).$$

The next step is to show that

$$(26) \quad \bigcup_{k \in \mathbb{N}} \left( S^1(-1, +1) \cap \left\{ \begin{array}{l} y_0 < 1 \\ y_1 < 1 \end{array} \right\} \right) - (k, 0) = \bigcup_{k \in \mathbb{N}} (S^1(-1, +1) - (k, 0)) \cap \left\{ \begin{array}{l} y_0 < 1 \\ y_1 < 1 \end{array} \right\}.$$

The inclusion  $\subset$  is clear. Assume now that  $a_0, a_1 \in \mathbb{R}$ ,  $k \in \mathbb{N}$  and  $|a_0 - a_1| < 1$  and  $a_0 - k < 1$ ,  $a_1 < 1$ . We have to find  $a'_0 \in \mathbb{R}$ , and  $k' \in \mathbb{N}$  such that

$$(27) \quad |a'_0 - a_1| < 1 \quad a'_0 < 1 \quad a'_0 - k' = a_0 - k.$$

First note that  $a_0 = a_0 - a_1 + a_1 < |a_0 - a_1| + a_1 < 2$ . If  $k = 0$  or  $a_0 < 1$ , then we put  $a'_0 = a_0$ ,  $k' = k$ . Thus, we can assume that  $k \geq 1$  and  $1 \leq a_0 < 2$ . Now  $a_1 < 1$  and  $|a_0 - a_1| < 1$  imply  $0 \leq a_1 < 1$ . It follows that  $-1 < -a_1 \leq a_0 - 1 - a_1 < 1$ , therefore we can put  $a'_0 = a_0 - 1$ ,  $k' = k - 1$ . Hence we obtain (26).

On the other hand by (13) and the already proven first equality in (18) we have

$$\bigcup_{k \in \mathbb{N}} S^1(-1, 1) - (k, 0) = \bigcup_{k \in \mathbb{N}} S^1(-1, 1) + (0, k) = S^1(-\infty, 1).$$

The latter equality and equalities (25), (26) imply (20) and the lemma follows.  $\square$

Putting (18) in (17) and then using (19) we obtain for the case  $n = 2$ :

$$(28) \quad \bigcup_{\mathbf{p} \in A} S^2(-\mathbf{1}, +\mathbf{1}) - \mathbf{p} = \bigcup_{\mathbf{p} \in A_0} S^2(-\infty, \mathbf{1}) - \mathbf{p} = \bigcup_{(0, p_1, p_2) \in A_0} \left\{ \begin{array}{l} y_0 - y_1 < 1 + p_1 \\ y_0 - y_2 < 1 + p_2 \\ y_1 - y_2 < 1 + p_2 - p_1 \end{array} \right\}.$$

Using the equality (28), the homeomorphism (10), and (11) we will prove the main result of this subsection:

**Proposition 2.7.** *Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. Let  $\mathcal{E} = (A_0, A_1, A_2)$  be a full exceptional collection, such that:*

$$(29) \quad \begin{aligned} 1 + \alpha &= \min\{i : \text{hom}^i(A_0, A_1) \neq 0\} \in \mathbb{Z} \\ 1 + \beta &= \min\{i : \text{hom}^i(A_0, A_2) \neq 0\} \in \mathbb{Z} \\ 1 + \gamma &= \min\{i : \text{hom}^i(A_1, A_2) \neq 0\} \in \mathbb{Z}. \end{aligned}$$

*Then the subset  $\Theta_{\mathcal{E}} \subset \text{Stab}(\mathcal{T})$  defined in (9) has the following description:*

$$(30) \quad \Theta_{\mathcal{E}} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} \phi^{\sigma}(A_0) - \phi^{\sigma}(A_1) < 1 + \alpha \\ \phi^{\sigma}(A_0) - \phi^{\sigma}(A_2) < 1 + \min\{\beta, \alpha + \gamma\} \\ \phi^{\sigma}(A_1) - \phi^{\sigma}(A_2) < 1 + \gamma \end{array} \right\}$$

and  $\Theta_\varepsilon$  is homeomorphic with the set  $\mathbb{R}_{>0}^3 \times \left\{ \begin{array}{l} y_0 - y_1 < 1 + \alpha \\ y_0 - y_2 < 1 + \min\{\beta, \alpha + \gamma\} \\ y_1 - y_2 < 1 + \gamma \end{array} \right\}$  by the map  $f_\varepsilon$  in (8) restricted to  $\Theta_\varepsilon$ . In particular  $\Theta_\varepsilon$  is contractible.

*Proof.* Given a family  $\mathcal{J}$  of the form:  $\mathcal{J} = \{I_{01} = (-\infty, u), I_{02} = (-\infty, v), I_{12} = (-\infty, w)\}$ , we write  $S \begin{pmatrix} -\infty, u \\ -\infty, v \\ -\infty, w \end{pmatrix}$  for  $S^2(\mathcal{J})$  throughout the proof. By (28), (11), and (10) the proof is reduced to showing that:

$$(31) \quad \bigcup_{(0, p_1, p_2) \in A_0} S \begin{pmatrix} -\infty, 1 + p_1 \\ -\infty, 1 + p_2 \\ -\infty, 1 + p_2 - p_1 \end{pmatrix} = S \begin{pmatrix} -\infty, 1 + \alpha \\ -\infty, 1 + \min\{\beta, \alpha + \gamma\} \\ -\infty, 1 + \gamma \end{pmatrix}.$$

From the definition of  $A_0$  in (16) and the definition of  $\alpha, \beta, \gamma$  one easily obtains:

$$(32) \quad (0, p_1, p_2) \in A_0 \Rightarrow p_1 \leq \alpha, p_2 \leq \min\{\beta, \alpha + \gamma\}; \quad (0, \alpha, \min\{\beta, \alpha + \gamma\}) \in A_0.$$

If  $u \leq u', v \leq v', w \leq w'$ , then  $S(-\infty, (u, v, w)) \subset S(-\infty, (u', v', w'))$ , hence by (32) we have:

$$(33) \quad \bigcup_{(0, p_1, p_2) \in A_0} S \begin{pmatrix} -\infty, 1 + p_1 \\ -\infty, 1 + p_2 \\ -\infty, 1 + p_2 - p_1 \end{pmatrix} = S \begin{pmatrix} -\infty, 1 + \alpha \\ -\infty, 1 + \min\{\beta, \alpha + \gamma\} \\ -\infty, 1 + \min\{\beta, \alpha + \gamma\} - \alpha \end{pmatrix} \cup \bigcup_{\left\{ \begin{array}{l} (0, p_1, p_2) \in A_0 : \\ p_2 - p_1 > \min\{\beta, \alpha + \gamma\} - \alpha \end{array} \right\}} S \begin{pmatrix} -\infty, 1 + p_1 \\ -\infty, 1 + p_2 \\ -\infty, 1 + p_2 - p_1 \end{pmatrix}.$$

Now we consider two cases.

If  $\min\{\beta, \alpha + \gamma\} = \alpha + \gamma$ , then  $\min\{\beta, \alpha + \gamma\} - \alpha = \gamma$  and  $(A_0, A_1[p_1], A_2[p_2])$  is not an Ext-collection for  $p_2 - p_1 > \gamma$  (since  $\text{hom}^{p_1 + \gamma + 1 - p_2}(A_1[p_1], A_2[p_2]) \neq 0, p_2 - p_1 - \gamma - 1 \geq 0$ ), hence the equality (33) reduces to (31).

If  $\min\{\beta, \alpha + \gamma\} = \beta < \alpha + \gamma$ , then  $\beta \leq \alpha - i + \gamma$  for  $i \leq \alpha + \gamma - \beta$  and hence

$$(34) \quad \{(0, \alpha - i, \beta) : 0 \leq i \leq \alpha + \gamma - \beta\} \subset A_0.$$

Furthermore, we claim that the equality (33) reduces to

$$(35) \quad \bigcup_{(0, p_1, p_2) \in A_0} S \begin{pmatrix} -\infty, 1 + p_1 \\ -\infty, 1 + p_2 \\ -\infty, 1 + p_2 - p_1 \end{pmatrix} = \bigcup_{i=0}^{\alpha + \gamma - \beta} S \begin{pmatrix} -\infty, 1 + \alpha - i \\ -\infty, 1 + \beta \\ -\infty, 1 + \beta - \alpha + i \end{pmatrix}.$$

Indeed, the first set of the union in (33) is the same as the first set of the union (35). Now assume that  $(0, p_1, p_2) \in A_0$  and  $p_2 - p_1 > \beta - \alpha$ , then  $\beta - \alpha < p_2 - p_1 \leq \gamma$ . Therefore for some  $1 \leq i \leq \gamma + \alpha - \beta$  we have  $p_2 - p_1 = \beta - \alpha + i$ . From (32) we have also  $p_2 \leq \beta$ , therefore  $p_1 = p_2 - \beta + \alpha - i \leq \alpha - i$ ,

and then  $S \begin{pmatrix} -\infty, 1 + p_1 \\ -\infty, 1 + p_2 \\ -\infty, 1 + p_2 - p_1 \end{pmatrix} \subset S \begin{pmatrix} -\infty, 1 + \alpha - i \\ -\infty, 1 + \beta \\ -\infty, 1 + \beta - \alpha + i \end{pmatrix}$  and we showed (35). The last step

of the proof is to show that

$$(36) \quad \bigcup_{i=0}^{\alpha+\gamma-\beta} S \begin{pmatrix} -\infty, 1+\alpha-i \\ -\infty, 1+\beta \\ -\infty, 1+\beta-\alpha+i \end{pmatrix} = S \begin{pmatrix} -\infty, 1+\alpha \\ -\infty, 1+\beta \\ -\infty, 1+\gamma \end{pmatrix}.$$

The inclusion  $\subset$  is clear. To show the inclusion  $\supset$ , assume that  $(a_0, a_1, a_2) \in \mathbb{R}^3$  and  $a_0 - a_1 < 1 + \alpha$ ,  $a_0 - a_2 < 1 + \beta$ ,  $a_1 - a_2 < 1 + \gamma$ .

If  $a_0 - a_1 < 1 + \alpha - (\alpha + \gamma - \beta) = 1 + \beta - \gamma$ , then by  $a_1 - a_2 < 1 + \gamma$  it follows that  $(a_0, a_1, a_2)$  is in the set with index  $i = \alpha + \gamma - \beta$  on the right-hand side.

It remains to consider the case, when  $1 + \alpha - i > a_0 - a_1 \geq 1 + \alpha - i - 1$  for some  $0 \leq i < \alpha + \gamma - \beta$ . Now  $(a_0, a_1, a_2)$  is in the set indexed by the given  $i$ . Indeed, now  $a_1 - a_0 \leq i - \alpha$  and by  $a_0 - a_2 < 1 + \beta$  we have  $a_1 - a_2 = a_1 - a_0 + a_0 - a_2 < 1 + \beta + i - \alpha$ .  $\square$

**2.4. More propositions used for gluing.** Since we will often use the notion of a  $\sigma$ -triple, for the sake of completeness we rewrite here [9, Definition 3.19] for triples (see also [9, Remark 3.31]):

**Definition 2.8.** An exceptional triple  $(A_0, A_1, A_2)$  is a  $\sigma$ -triple iff the following conditions hold:

(a)  $\text{hom}^{\leq 0}(A_i, A_j) = 0$  for  $i \neq j$ ; (b)  $\{A_i\}_{i=0}^2 \subset \sigma^{ss}$ ; (c)  $\{\phi(A_i)\}_{i=0}^2 \subset (t, t+1)$  for some  $t \in \mathbb{R}$ .

We enhance now Proposition 2.2 for the case  $n = 2$ :

**Proposition 2.9.** Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. Let  $\mathcal{E} = (A_0, A_1, A_2)$ ,  $\alpha, \beta, \gamma$  be as in Proposition 2.7. Let  $\sigma \in \Theta_{\mathcal{E}}$  (hence we have the inequalities in (30)).

(a) If  $\phi^{\sigma}(A_0) \geq \phi^{\sigma}(A_1[\alpha])$ , then  $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$ , where  $\mathcal{A}$  is the extension closure of  $(A_0, A_1[\alpha])$ .

(b) If  $\phi^{\sigma}(A_1) \geq \phi^{\sigma}(A_2[\gamma])$ , then  $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$ , where  $\mathcal{A}$  is the extension closure of  $(A_1, A_2[\gamma])$ .

*Proof.* If an equality holds in (a) or (b), then we have  $\mathcal{A} \subset \mathcal{P}(t)$  for some  $t \in \mathbb{R}$  and the Proposition follows. Hence we can assume that we have a proper inequality in both the cases.

(a) By the definition of  $\Theta_{\mathcal{E}}$  in (9) and [9, Corollary 3.20] we see that  $(A_0[l], A_1[i], A_2[j])$  is a  $\sigma$ -triple for some  $l, i, j \in \mathbb{Z}$ . We can assume<sup>3</sup>  $l = 0$  and then  $\text{hom}^{\leq 0}(A_0, A_1[i]) = 0$  and  $|\phi(A_0) - \phi(A_1[i])| < 1$ . From the definition of  $\alpha$  we see that  $i \leq \alpha$ . Actually we must have  $i = \alpha$ , otherwise the given inequality  $\phi(A_0) - \phi(A_1[\alpha]) > 0$  implies  $\phi(A_0) - \phi(A_1[i]) > 1$ , which is a contradiction. Thus  $(A_0, A_1[\alpha], A_2[j])$  is a  $\sigma$ -triple for some  $j \in \mathbb{Z}$ . Now we apply Proposition 2.2.

(b) In this case we shift the given triple to a  $\sigma$ -triple of the form  $(A_0[l], A_1, A_2[j])$  for some  $l, j \in \mathbb{Z}$ , in particular we have  $\text{hom}^{\leq 0}(A_1, A_2[j]) = 0$  and  $|\phi(A_1) - \phi(A_2[j])| < 1$ . From the definition of  $\gamma$  and the given inequality  $\phi(A_1) - \phi(A_2[\gamma]) > 0$  it follows that  $j = \gamma$ . Thus  $(A_0[l], A_1, A_2[\gamma])$  is a  $\sigma$ -triple for some  $l \in \mathbb{Z}$ . Now we apply Proposition 2.2.  $\square$

**Proposition 2.10.** Let  $\mathcal{T}$  has the property that for each exceptional triple  $(A_0, A_1, A_2)$  and any two  $0 \leq i < j \leq 2$  there exists unique  $k \in \mathbb{Z}$  satisfying  $\text{hom}^k(A_i, A_j) \neq 0$ . Let  $\mathcal{E} = (A_0, A_1, A_2)$  be a full exceptional collection in  $\mathcal{T}$ .

Let  $R_0(\mathcal{E}) = (A_1, R_{A_1}(A_0), A_2)$ ,  $L_0(\mathcal{E}) = (L_{A_0}(A_1), A_0, A_2)$ ,  $R_1(\mathcal{E}) = (A_0, A_2, R_{A_2}(A_1))$ ,  $L_1(\mathcal{E}) = (A_0, L_{A_1}(A_2), A_1)$  be the triples obtained by a single mutation applied to  $\mathcal{E}$ .<sup>4</sup> Then the four intersections  $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$ ,  $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})}$ ,  $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})}$ ,  $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$  are all contractible and non-empty.

<sup>3</sup>note that  $(A_0, A_1[i], A_2[j])$  is a  $\sigma$ -triple iff  $(A_0[k], A_1[i+k], A_2[j+k])$  is a  $\sigma$ -triple

<sup>4</sup> Recall that for any exceptional pair  $(A, B)$  the exceptional objects  $L_A(B)$  and  $R_B(A)$  are determined by the triangles  $L_A(B) \rightarrow \text{Hom}^*(A, B) \otimes A \xrightarrow{ev_{A,B}^*} B$ ;  $A \xrightarrow{coev_{A,B}^*} \text{Hom}^*(A, B) \otimes B \rightarrow R_B(A)$  and that  $(L_A(B), A)$ ,  $(B, R_B(A))$  are exceptional pairs.

*Proof.* Since  $\mathcal{E} \sim \mathcal{E}'$  implies  $\Theta_{\mathcal{E}} = \Theta_{\mathcal{E}'}$ ,  $R_i(\mathcal{E}) \sim R_i(\mathcal{E}')$ ,  $L_i(\mathcal{E}) \sim L_i(\mathcal{E}')$ , we can assume that  $l = \text{hom}^1(A_0, A_1) > 0$ ,  $p = \text{hom}^1(A_1, A_2) > 0$ . By the assumptions on  $\mathcal{T}$  the other degrees are zero and it follows that the integers  $\alpha, \gamma$  defined in (29) vanish and from Proposition 2.7 we get:

$$(37) \quad \Theta_{\mathcal{E}} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1 \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\} \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < 1 \end{array} \right\}.$$

We start with the intersection  $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$ . Let us denote  $X = R_{A_1}(A_0)[-1]$ . Let  $\alpha', \beta', \gamma'$  be the integers corresponding to the triple  $(A_1, X, A_2)$  used in Proposition 2.7. We have  $1 + \beta' = \min\{k : \text{hom}^k(A_1, A_2) \neq 0\} = 1 + \gamma = 1$ , hence  $\beta' = 0$ . On the other hand from the definition of  $R_{A_1}(A_0)$  we have a triangle

$$(38) \quad A_1^{\oplus l} \rightarrow X \rightarrow A_0 \rightarrow A_1^{\oplus l}[1]$$

and it follows that  $\text{hom}(A_1, X) \neq 0$ , hence  $\alpha' = -1$ . We apply Proposition 2.7 to the triple  $(A_1, X, A_2)$  and obtain (note that  $1 + \min\{\beta', \alpha' + \gamma'\} = 1 + \min\{0, \gamma' - 1\} = \min\{1, \gamma'\}$ )

$$(39) \quad \Theta_{R_0(\mathcal{E})} = \Theta_{(A_1, X, A_2)} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \begin{array}{ll} A_1 \in \sigma^{ss} & \phi^\sigma(A_1) - \phi^\sigma(X) < 0 \\ X \in \sigma^{ss} & \text{and } \phi^\sigma(A_1) - \phi^\sigma(A_2) < \min\{1, \gamma'\} \\ A_2 \in \sigma^{ss} & \phi^\sigma(X) - \phi^\sigma(A_2) < 1 + \gamma' \end{array} \right\}.$$

From the definition of  $\beta, \gamma$  we have  $0 = \text{hom}^{\leq \min\{\beta, \gamma\}}(A_0, A_2) = \text{hom}^{\leq \min\{\beta, \gamma\}}(A_1, A_2)$ , and then the triangle (38) implies that  $\text{hom}^{\leq \min\{\beta, \gamma\}}(X, A_2) = 0$ , it follows that

$$(40) \quad \min\{\beta, \gamma\} = \min\{\beta, 0\} \leq \gamma'.$$

Assume that  $\sigma \in \Theta_{(A_1, X, A_2)} \cap \Theta_{\mathcal{E}}$ . Then  $A_0, A_1, A_2, X$  are all semi-stable and  $\phi(A_1) < \phi(X)$ .<sup>5</sup> It is easy to show that  $\text{hom}(X, A_0) \neq 0$  (using the triangle (38)), hence  $\phi(X) \leq \phi(A_0)$  and therefore  $\phi(A_1) < \phi(A_0)$ , and we obtain the inclusion  $\subset$  in the following formula (the third inequality in this formula is the second in (39), the other inequalities are in (37) together with  $\phi(A_1) < \phi(A_0)$ )

$$(41) \quad \Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} 0 < \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1 \\ \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\} \\ \phi^\sigma(A_1) - \phi^\sigma(A_2) < \min\{\gamma', 1\} \end{array} \right\}.$$

We show now the inclusion  $\supset$ . Assume that  $\mathcal{E} \subset \sigma^{ss}$  and that the inequalities on the right hand side of (41) hold. In particular the inequalities in (37) hold, hence we have  $\sigma \in \Theta_{\mathcal{E}}$  and  $\phi^\sigma(A_0) > \phi^\sigma(A_1)$ . Proposition 2.9 (a) ensures  $X \in \sigma^{ss}$  and by (38) we get  $\text{hom}(A_1, X) \neq 0$ ,  $\text{hom}(X, A_2) \neq 0$ , hence

$$(42) \quad X \in \sigma^{ss} \quad \phi(A_1) \leq \phi(X) \leq \phi(A_0) \quad Z(X) = lZ(A_1) + Z(A_0).$$

Using (86) and  $0 < \phi^\sigma(A_0) - \phi^\sigma(A_1) < 1$  we see that  $Z(A_1), Z(A_0)$  are not collinear (see Definition 3.16), therefore  $Z(X) = lZ(A_1) + Z(A_0)$  is collinear neither with  $Z(A_1)$  nor with  $Z(A_0)$ . Now we apply (86) again and by (42) we obtain  $\phi(A_1) < \phi(X) < \phi(A_0)$ . In particular, we obtain the first inequality in (39). The second inequality in (39) is the same as the third inequality of (41). From  $\phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \min\{\beta, 0\}$  and (40) we get  $\phi(X) - \phi^\sigma(A_2) < \phi^\sigma(A_0) - \phi^\sigma(A_2) < 1 + \gamma'$ , hence the third inequality in (39) is verified also. Thus we showed (41). This equality implies that the set  $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$  is contractible. Indeed, we have a homeomorphism  $f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \rightarrow f_{\mathcal{E}}(\Theta_{\mathcal{E}})$  (see (10), (8)).

<sup>5</sup>We omit sometimes the superscript  $\sigma$  in expressions like  $\phi^\sigma(X)$  and write just  $\phi(X)$ .

The proved equality (41) shows that:  $f_{\mathcal{E}}(\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}) = \mathbb{R}_{>0}^3 \times \left\{ \begin{array}{l} 0 < \phi_0 - \phi_1 < 1 \\ \phi_0 - \phi_2 < 1 + \min\{\beta, 0\} \\ \phi_1 - \phi_2 < \min\{\gamma', 1\} \end{array} \right\}$ ,

hence  $\Theta_{\mathcal{E}} \cap \Theta_{R_0(\mathcal{E})}$  is contractible.

Next, we consider the intersection  $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$ , where  $L_1(\mathcal{E}) = (A_0, L_{A_1}(A_2), A_1)$ .

Let us denote  $Y = L_{A_1}(A_2)[1]$ . Let  $\alpha', \beta', \gamma'$  be the integers corresponding to the triple  $(A_0, Y, A_1)$ . Obviously  $\beta' = \alpha = 0$ . From the definition of  $L_{A_1}(A_2)$  we have a triangle

$$(43) \quad A_2 \rightarrow Y \rightarrow A_1^{\oplus p} \rightarrow A_2[1]$$

and it follows that  $\text{hom}(Y, A_1) \neq 0$ , hence  $\gamma' = -1$ . Proposition 2.7 applied to the triple  $(A_0, Y, A_1)$  results in the equality (note that  $1 + \min\{0', \alpha' - 1\} = \min\{1, \alpha'\}$ )

$$(44) \quad \Theta_{L_1(\mathcal{E})} = \Theta_{(A_0, Y, A_1)} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \begin{array}{ll} A_0 \in \sigma^{ss} & \phi^{\sigma}(A_0) - \phi^{\sigma}(Y) < 1 + \alpha' \\ Y \in \sigma^{ss} & \text{and } \phi^{\sigma}(A_0) - \phi^{\sigma}(A_1) < \min\{1, \alpha'\} \\ A_1 \in \sigma^{ss} & \phi^{\sigma}(Y) - \phi^{\sigma}(A_1) < 0 \end{array} \right\}.$$

From the definition of  $\alpha, \beta$  for the initial sequence  $\mathcal{E}$  we have  $0 = \text{hom}^{\leq \min\{\alpha, \beta\}}(A_0, A_1)$ ,  $0 = \text{hom}^{\leq \min\{\alpha, \beta\}}(A_0, A_2)$ , and then the triangle (43) implies that  $\text{hom}^{\leq \min\{\alpha, \beta\}}(A_0, Y) = 0$ , it follows that

$$(45) \quad \min\{\alpha, \beta\} = \min\{0, \beta\} \leq \alpha'.$$

Assume that  $\sigma \in \Theta_{(A_0, Y, A_1)} \cap \Theta_{\mathcal{E}}$ . Then  $A_0, A_1, A_2, Y$  are all semi-stable and by (44)  $\phi(Y) < \phi(A_1)$ . The triangle (43) implies  $\text{hom}(A_2, Y) \neq 0$ , hence  $\phi(A_2) \leq \phi(Y) < \phi(A_1)$ . Combining this inequality with the inequalities in (44), (37) we obtain the inclusion  $\subset$  in the following formula:

$$(46) \quad \Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \subset \sigma^{ss} \text{ and } \begin{array}{l} \phi^{\sigma}(A_0) - \phi^{\sigma}(A_1) < \min\{1, \alpha'\} \\ \phi^{\sigma}(A_0) - \phi^{\sigma}(A_2) < 1 + \min\{\beta, 0\} \\ 0 < \phi^{\sigma}(A_1) - \phi^{\sigma}(A_2) < 1 \end{array} \right\}.$$

To show the inclusion  $\supset$ , assume that  $\mathcal{E} \subset \sigma^{ss}$  and that the inequalities on the right hand side of (46) hold. In particular, we have  $\sigma \in \Theta_{\mathcal{E}}$  (see (37)). It remains to show that  $Y \in \sigma^{ss}$  and that the inequalities in (44) hold. From Proposition 2.9 (b) and  $\sigma \in \Theta_{\mathcal{E}}$ ,  $\phi^{\sigma}(A_1) > \phi^{\sigma}(A_2)$  we obtain  $Y \in \sigma^{ss}$ . The triangle (43) implies

$$(47) \quad Y \in \sigma^{ss} \quad \phi(A_2) \leq \phi(Y) \leq \phi(A_1) \quad Z(Y) = pZ(A_1) + Z(A_2).$$

By similar arguments as in the previous case, using (86),  $0 < \phi^{\sigma}(A_1) - \phi^{\sigma}(A_2) < 1$  and (47) one shows that  $\phi(A_2) < \phi(Y) < \phi(A_1)$ . In particular, we obtain the third inequality in (44). The second inequality in (44) is the same as the first inequality of (46). From  $\phi^{\sigma}(A_0) - \phi^{\sigma}(A_2) < 1 + \min\{\beta, 0\}$  and (45) we get  $\phi(A_0) - \phi^{\sigma}(Y) < \phi^{\sigma}(A_0) - \phi^{\sigma}(A_2) < 1 + \alpha'$  and the first inequality in (46) is verified also. Thus we showed (46). As in the previous case this implies that  $\Theta_{\mathcal{E}} \cap \Theta_{L_1(\mathcal{E})}$  is contractible.

Finally, recall that  $\mathcal{E} \sim R_0(L_0(\mathcal{E}))$ , therefore  $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})} = \Theta_{R_0(L_0(\mathcal{E}))} \cap \Theta_{L_0(\mathcal{E})}$  and by the already proved first case we see that  $\Theta_{\mathcal{E}} \cap \Theta_{L_0(\mathcal{E})}$  is contractible. For the case  $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})}$  we have  $\Theta_{\mathcal{E}} \cap \Theta_{R_1(\mathcal{E})} = \Theta_{L_1(R_1\mathcal{E})} \cap \Theta_{R_1(\mathcal{E})}$  and contractibility follows from a previous case. The Proposition is proved.  $\square$

Propositions 2.9 and 2.2 ensure semi-stability of certain exceptional objects. The following two lemmas are similar in that respect and will be used later, when we analyze the intersections of the form  $\Theta_{\mathcal{E}_1} \cap \Theta_{\mathcal{E}_2}$ , when  $\mathcal{E}_2$  is obtained from  $\mathcal{E}_1$  by more than one and different mutations.

**Lemma 2.11.** *Let  $\mathcal{T} = D^b(\mathcal{A})$ , where  $\mathcal{A}$  is a hereditary abelian hom-finite category, and let for any two exceptional objects  $E, F \in \mathcal{T}_{exc}$  there exists at most one  $k \in \mathbb{Z}$  satisfying  $\text{hom}^k(E, F) \neq 0$ .*

*Let  $(A_0, A_1, A_2)$  be a full Ext-exceptional (“Ext-” means that it satisfies (a) in Definition 2.8) collection in  $\mathcal{T}$ , such that  $\text{hom}^1(A_0, A_2) = 0$  and  $A_0, A_1, A_2$  are semistable. Let  $X, Y$  be exceptional objects in  $\mathcal{T}$  for which we have a diagram of distinguished triangles, where all arrows are non-zero:*

$$(48) \quad \begin{array}{ccccccc} 0 & \xrightarrow{\quad} & A_2 & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ & & A_2 & & A_1 & & A_0 \end{array}.$$

(a) *If we have the following system of inequalities:*

$$\begin{aligned} \phi(A_0) - 1 < \phi(A_1) < \phi(A_0), \quad \phi(A_0) - 1 < \phi(A_2) < \phi(A_0) \\ \arg(\phi(A_0) - 1, \phi(A_0))(Z(A_0) + Z(A_1)) > \phi(A_2) \end{aligned},$$

*then  $Y \in \sigma^{ss}$  and  $\phi(Y) < \phi(A_0)$ .*

(b) *If we have  $\phi(A_2) < \phi(A_1) \leq \phi(A_0) < \phi(A_2) + 1$ , then  $Y \in \sigma^{ss}$  and  $\phi(Y) < \phi(A_0)$ .*

*Proof.* We note first some vanishings. From the given diagram it follows that  $\text{hom}(Y, A_0) \neq 0$  and  $\text{hom}(X, A_1) \neq 0$ . Since  $X, Y$  are also exceptional objects, from [9, Lemma 9.1] and the hereditariness of  $\mathcal{A}$  it follows that  $\text{hom}(A_0, Y) = \text{hom}(A_1, X) = 0$ . On the other hand  $\text{hom}(A_1, Y) = \text{hom}(A_1, X)$  (follows by applying  $\text{hom}(A_1, \_)$  to the last triangle and using  $\text{hom}^*(A_1, A_0) = 0$ ). Thus, we obtain

$$(49) \quad \text{hom}(A_0, Y) = \text{hom}(A_1, Y) = 0.$$

Since  $(A_0, A_1, A_2)$  is an Ext-exceptional collection, its extension closure is a heart of a bounded t-structure ([16, Lemma 3.14]), furthermore this heart is of finite length and  $(A_0, A_1, A_2)$  are the simple objects in it. Let us denote for simplicity  $t = \phi(A_0)$  (in case (a)) or  $t - 1 = \phi(A_2)$  (in case (b)). In both the cases from the given inequalities and since  $\mathcal{P}(t - 1, t]$  is also a heart, it follows that the extension closure of  $(A_0, A_1, A_2)$  is exactly  $\mathcal{P}(t - 1, t]$ . Now (48) can be considered as the Jordan-Hölder filtration of  $Y$  in the abelian category  $\mathcal{P}(t - 1, t]$  and the composition factors of  $Y$  are  $\{A_0, A_1, A_2\}$ .

Suppose that  $Y \notin \sigma^{ss}$ . From Remark 2.1 (b) there exists  $Y' \in \mathcal{P}(t - 1, t]$  and a non-trivial monic arrow  $Y' \rightarrow Y$ , s. t.  $\arg_{(t-1, t]}(Z(Y')) > \arg_{(t-1, t]}(Z(Y))$ .

We have  $Z(Y) = Z(A_0) + Z(A_1) + Z(A_2)$  and one can show that the given inequalities in either case (a) or (b) imply that

$$(50) \quad \arg_{(t-1, t]}(Z(Y)) > \arg_{(t-1, t]}(Z(A_2)), \quad \arg_{(t-1, t]}(Z(Y)) > \arg_{(t-1, t]}(Z(A_2) + Z(A_1)).$$

Since  $Y'$  is a subobject of  $Y$ , the composition factors of  $Y'$  in  $\mathcal{P}(t - 1, t]$  are subset of  $\{A_0, A_1, A_2\}$ . The cases  $Y' \cong A_0$ ,  $Y' \cong A_1$  are excluded by (49). The case  $Y' \cong A_2$  is excluded by the first inequality in (50) and the condition  $\arg_{(t-1, t]}(Z(Y')) > \arg_{(t-1, t]}(Z(Y))$ . Since  $Y'$  is a proper subobject of  $Y$  we reduce to the case when  $Y'$  has two composition factors (two different elements of the set  $\{A_0, A_1, A_2\}$ ). Using (49) again we reduce to the following options for a Jordan Hölder filtration

$$(51) \quad 0 \rightarrow A_2 \rightarrow Y' \rightarrow A_1 \rightarrow 0 \quad 0 \rightarrow A_2 \rightarrow Y' \rightarrow A_0 \rightarrow 0.$$

In the first case we have  $Z(Y') = Z(A_2) + Z(A_1)$  which contradicts the second inequality on (50). In the second case we have a distinguished triangle  $A_2 \rightarrow Y' \rightarrow A_0 \rightarrow A_2[1]$  in  $\mathcal{T}$ , and from the given vanishing  $\text{hom}^1(A_0, A_2) = 0$  it follows  $Y' \cong A_0 \oplus A_2$ , which contradicts (49). So we proved

$Y \in \sigma^{ss}$ . The inequality  $\phi(Y) < \phi(A_0)$  (in either case (a) or (b)) follows from the given inequalities and  $Z(Y) = Z(A_0) + Z(A_1) + Z(A_2)$ . The lemma is proved.  $\square$

**Lemma 2.12.** *Let  $\mathcal{T} = D^b(\mathcal{A})$  be as in Lemma 2.11.*

*Let  $(A_0, A_1, A_2)$  be a full Ext-exceptional collection<sup>6</sup> in  $\mathcal{T}$  such that  $A_0, A_1, A_2$  are semistable. Let  $Y$  be an exceptional object in  $\mathcal{T}$  for which we have a triangle, where all arrows are non-zero:*

$$(52) \quad \begin{array}{ccc} A_2 & \xrightarrow{\quad} & Y \\ & \searrow \quad \swarrow & \\ & A_0 & \end{array}.$$

*If one of the two systems:  $\phi(A_2) < \phi(A_0) < \phi(A_2) + 1$  or  $\phi(A_0) - 1 < \phi(A_1) < \phi(A_0)$  holds, then we have:  $Y \in \sigma^{ss}$ ,  $\phi(Y) = \arg_{(\phi(A_2), \phi(A_2)+1)}(Z(A_0) + Z(A_2)) = \arg_{(\phi(A_0)-1, \phi(A_0))}(Z(A_0) + Z(A_2))$  and  $\phi(A_2) < \phi(Y) < \phi(A_0)$ .*

*Proof.* Since  $Y, A_0$  are exceptional objects and  $\text{hom}(Y, A_0) \neq 0$ , from [9, Lemma 9.1] and the hereditariness of  $\mathcal{A}$  it follows that  $\text{hom}(A_0, Y) = 0$ . Due to the given inequalities, in both the cases we can choose  $t$  so that  $\phi(A_0), \phi(A_1), \phi(A_2) \in (t-1, t]$ . By the same arguments as in the previous lemma, one sees that the extension closure of  $(A_0, A_1, A_2)$  is  $\mathcal{P}(t-1, t]$  and that this is an abelian category of finite length with simple objects  $A_0, A_1, A_2$ . Now (52) can be considered as the Jordan-Hölder filtration of  $Y$  in the abelian category  $\mathcal{P}(t-1, t]$  and the composition factors of  $Y$  are  $\{A_0, A_2\}$ .

We have  $Z(Y) = Z(A_0) + Z(A_2)$  and the given inequalities (in either case) imply that:

$$(53) \quad \begin{aligned} \phi(A_2) &= \arg_{(t-1, t]}(Z(A_2)) < \arg_{(t-1, t]}(Z(Y)) = \arg_{(\phi(A_0)-1, \phi(A_0))}(Z(A_0) + Z(A_2)) \\ &= \arg_{(\phi(A_2), \phi(A_2)+1)}(Z(A_0) + Z(A_2)) < \arg_{(t-1, t]}(Z(A_0)) = \phi(A_0). \end{aligned}$$

Suppose that  $Y \notin \sigma^{ss}$ . From Remark 2.1 (b) it follows that there exists  $Y' \in \mathcal{P}(t-1, t]$  and a non-trivial monic arrow  $Y' \rightarrow Y$  in  $\mathcal{P}(t-1, t]$ , s. t.  $\arg_{(t-1, t]}(Z(Y')) > \arg_{(t-1, t]}(Z(Y))$ . Since the composition factors of  $Y$  are  $\{A_0, A_2\}$  and  $Y'$  is a non-zero proper sub-object of  $Y$ , then we have  $Y' \cong A_2$  or  $Y' \cong A_0$ . The case  $Y' \cong A_0$ , is excluded by  $\text{hom}(A_0, Y) = 0$ . The case  $Y' \cong A_2$  is excluded by (53) and the condition  $\arg_{(t-1, t]}(Z(Y')) > \arg_{(t-1, t]}(Z(Y))$ . So we proved  $Y \in \sigma^{ss}$ .

By  $Y \in \mathcal{P}(t, t+1]$  it follows that  $\phi(Y) = \arg_{(t-1, t]}(Z(Y))$ . Now the lemma follows from (53).  $\square$

### 3. THE EXCEPTIONAL OBJECTS IN $D^b(Q)$

From now on we fix  $\mathcal{T} = D^b(\text{Rep}_k(Q))$ , where  $Q$  is the affine quiver in figure (1).

In this Section we organize in a better way the data about  $\{\text{Hom}(X, Y), \text{Ext}^1(X, Y)\}_{X \in \mathcal{T}_{exc}}$  obtained in [9, Section 2]. In Subsection 3.3 are given some observations about the behavior of the vectors  $\{Z(X)\}_{X \in \mathcal{T}_{exc}} \subset \mathbb{C}$ , which will be helpful when we analyze the intersections of the form  $\Theta_{\mathcal{E}_1} \cap \Theta_{\mathcal{E}_2}$  in the next sections.

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<sup>6</sup> “Ext-” means that it satisfies (a) in Definition 2.8



We start by recalling the classification of the exceptional objects in  $\text{Rep}_k(Q)$  obtained in [9]. Let us denote for any  $m \geq 1$ :

$$\begin{aligned} \pi_+^m : k^{m+1} &\rightarrow k^m, & \pi_-^m : k^{m+1} &\rightarrow k^m, & j_+^m : k^m &\rightarrow k^{m+1}, & j_-^m : k^m &\rightarrow k^{m+1} \\ \pi_+^m(a_1, a_2, \dots, a_m, a_{m+1}) &= (a_1, a_2, \dots, a_m) & \pi_-^m(a_1, a_2, \dots, a_m, a_{m+1}) &= (a_2, \dots, a_m, a_{m+1}) \\ j_+^m(a_1, a_2, \dots, a_m) &= (a_1, a_2, \dots, a_m, 0) & j_-^m(a_1, a_2, \dots, a_m) &= (0, a_1, \dots, a_m). \end{aligned}$$

In [9] was shown that:

**Proposition 3.1.** [9, Proposition 2.2] *The exceptional objects up to isomorphism in  $\text{Rep}_k(Q)$  are  $(m = 0, 1, 2, \dots)$*

$$\begin{array}{ccccc} E_1^m = & \begin{array}{ccc} & k^m & \\ \pi_+^m \nearrow & & \searrow Id \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} & E_2^m = & \begin{array}{ccc} & k^{m+1} & \\ j_+^m \nearrow & & \searrow Id \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} & E_3^m = & \begin{array}{ccc} & k^{m+1} & \\ j_+^m \nearrow & & \searrow j_-^m \\ k^m & \xrightarrow{Id} & k^m \end{array} \\ E_4^m = & \begin{array}{ccc} & k^m & \\ \pi_+^m \nearrow & & \searrow \pi_-^m \\ k^{m+1} & \xrightarrow{Id} & k^{m+1} \end{array} & M = & \begin{array}{ccc} & 0 & \\ & \nearrow & \searrow \\ 0 & \xrightarrow{\quad} & k \end{array} & M' = & \begin{array}{ccc} & k & \\ Id \nearrow & & \searrow \\ k & \xrightarrow{\quad} & 0 \end{array} \end{array}$$

We denote by  $K(\mathcal{T})$  the Grothendieck group of  $\mathcal{T}$ . For  $X \in \mathcal{T}$  we denote by  $[X] \in K(\mathcal{T})$  the corresponding equivalence class in  $K(\mathcal{T})$ . From Proposition 3.1 it follows:

**Corollary 3.2.** *Let us denote  $\delta = [E_1^0] + [E_3^0] + [M] \in K(\mathcal{T})$ . We have the following equalities in  $K(\mathcal{T})$ :*

$$(54) \quad \delta = [E_1^0] + [E_3^0] + [M] = [E_1^0] + [E_2^0] = [E_3^0] + [E_4^0] = [M] + [M']$$

$$(55) \quad [E_1^m] = m\delta + [E_1^0] = (m+1)\delta - [E_2^0] \quad [E_2^m] = m\delta + [E_2^0] = (m+1)\delta - [E_1^0]$$

$$(56) \quad [E_3^m] = m\delta + [E_3^0] = (m+1)\delta - [E_4^0] \quad [E_4^m] = m\delta + [E_4^0] = (m+1)\delta - [E_3^0]$$

$$(57) \quad [E_1^m] + [M] = [E_4^m] \quad [E_3^m] + [M] = [E_2^m] \quad [E_4^m] + [M'] = [E_1^{m+1}] \quad [E_2^m] + [M'] = [E_3^{m+1}].$$

### 3.1. The two orbits of 2-Kronecker pairs in $D^b(Q)$ .

In Propositions 2.2 and 2.9 were discussed exceptional pairs  $(E, F)$  with  $\text{hom}^{\leq 0}(E, F) = 0$  and  $\text{hom}^1(E, F) = l \neq 0$ , and their extension closures. We call such a pair *l-Kronecker pair*. Kronecker pairs were used in [8] for studying the density of the set of phases of Bridgeland stability conditions. In [8, Corollary 3.31] was shown that for any affine acyclic quiver  $A$  (like the quiver  $Q$  in figure (1)) only 1- and 2-Kronecker pairs can appear in  $D^b(A)$ . In this subsection we give some comments on the 1- and 2-Kronecker pairs in  $D^b(Q)$ , which will be useful later when we apply Propositions 2.2, 2.9 and Lemmas 2.11, 2.12.

From [9, Remark 2.11] we see that the 2-Kronecker pairs in  $D^b(Q)$  up to shifts are:

$$(58) \quad \mathfrak{P}_{12} = \{(E_1^{m+1}, E_1^m[-1]), (E_1^0, E_2^0), (E_2^m, E_2^{m+1}[-1]) : m \in \mathbb{N}\}$$

$$(59) \quad \mathfrak{P}_{43} = \{(E_4^{m+1}, E_4^m[-1]), (E_4^0, E_3^0), (E_3^m, E_3^{m+1}[-1]) : m \in \mathbb{N}\}.$$

Recall that the Braid group on two strings  $B_2 \cong \mathbb{Z}$  acts on the set of equivalence classes of exceptional pairs in  $\mathcal{T}$  (here we take the equivalence  $\sim$  explained in **Some notations** and it is clear when a given equivalence class w.r.  $\sim$  will be called a 2-Kronecker pair). Using [9, Corollary 2.10] and the list of triples in [9, Corollary 2.9] one can show that the set of 2-Kronecker pairs is invariant under this action of  $B_2$  and this action on the 2-Kronecker pairs has two orbits. They are (58) and (59).

We will describe now the sets  $\mathcal{T}_{exc} \cap \mathcal{A}$ , up to isomorphism, where  $\mathcal{A}$  is the extension closure in  $\mathcal{T}$  of a 2-Kronecker pair. This will be helpful later (e. g. when we apply Propositions 2.2 and 2.9). We note first a simple lemma (in which  $Rep_k(Q)$  can be any hereditary category):

**Lemma 3.3.** *Let  $A, B \in Rep_k(Q)$ , let  $\mathcal{C}$  be the extension closure of  $A, B[-1]$  in  $D^b(Rep_k(Q)) = \mathcal{T}$ . Then any  $X \in \mathcal{C} \cap \mathcal{T}_{exc}$  has the form  $X'[i]$ , where  $X' \in Rep_k(Q)_{exc}$  and  $i \in \{0, -1\}$ .*

*Proof.* Since  $Rep_k(Q)$  is hereditary, any object  $X \in D^b(Rep_k(Q))$  decomposes as follows  $X \cong \bigoplus_{i \in \mathbb{Z}} H^i(X)[-i]$ , where  $H^i : \mathcal{T} \rightarrow Rep_k(Q)$  are the cohomology functors.

Since  $A, B \in Rep_k(Q)$ , it follows that  $H^i(A) = H^i(B[-1]) = 0$  for each  $i \neq \{0, 1\}$ . The functors  $H^i : \mathcal{T} \rightarrow Rep_k(Q)$  map triangles to short exact sequences (see e.g. [10]), therefore  $H^i(X) = 0$  for any  $X \in \mathcal{C}$  and any  $i \neq \{0, 1\}$ . By the first paragraph of the proof we see that each  $X \in \mathcal{C}$  has the form  $X' \oplus X''[-1]$  with  $X', X'' \in Rep_k(Q)$ . Finally, if  $X \in \mathcal{C} \cap \mathcal{T}_{exc}$ , then  $X$  is indecomposable in  $\mathcal{T}$ , hence either  $X \cong X'$  or  $X \cong X'[-1]$  for some  $X' \in Rep_k(Q)$  and obviously  $X'$  is also exceptional, i. e.  $X' \in Rep_k(Q)_{exc}$ .  $\square$

**Lemma 3.4.** *Let  $(U, V)$  be one of the 2-Kronecker pairs given in (58) or (59). Let  $\mathcal{A}$  be its extension closure in  $\mathcal{T}$ . Then representatives of the iso-classes of objects in  $\mathcal{A} \cap \mathcal{T}_{exc}$  are:*

$(U, V) =$	$(E_{1/4}^{m+1}, E_{1/4}^m[-1])$	$(E_{1/4}^0, E_{2/3}^0)$	$(E_{2/3}^m, E_{2/3}^{m+1}[-1])$
$\mathcal{A} \cap \mathcal{T}_{exc} =$	$\left\{ \begin{array}{ll} E_{1/4}^n[-1] & 0 \leq n \leq m \\ E_{1/4}^n & n \geq m+1 \\ E_{2/3}^n & n \in \mathbb{N} \end{array} \right\}$	$\left\{ \begin{array}{ll} E_{1/4}^n & n \in \mathbb{N} \\ E_{2/3}^n & n \in \mathbb{N} \end{array} \right\}$	$\left\{ \begin{array}{ll} E_{2/3}^n & 0 \leq n \leq m \\ E_{2/3}^n[-1] & n \geq m+1 \\ E_{1/4}^n[-1] & n \in \mathbb{N} \end{array} \right\}$

where the subscript in the table is either everywhere the first or everywhere the second.

*Proof.* We show the case when the subscript is everywhere the first (i. e. the pairs in (58)), the other case is analogous. From [8, Lemma 3.22] we have that  $\mathcal{A}$  is a bounded t-structure in  $\langle U, V \rangle$  and we have also an equivalence of abelian categories

$$(60) \quad F : \mathcal{A} \rightarrow Rep_k(K(2)) \quad F(U) = k \rightrightarrows 0, \quad F(V) = 0 \rightrightarrows k.$$

Using the facts that  $\mathcal{A}$  is a bounded t-structure and that  $F$  is equivalence, one can show that if  $X \in \mathcal{A} \cap \mathcal{T}_{exc}$ , then  $F(X) \in Rep_k(K(2))_{exc}$ . Furthermore, since  $\mathcal{T} = D^b(Rep_k(Q))$  and  $Rep_k(Q)$  is a hereditary abelian category, it is easy to show that:

$$(61) \quad X \in \mathcal{A} \cap \mathcal{T}_{exc} \Leftrightarrow F(X) \in Rep_k(K(2))_{exc}.$$

As in the proof of [9, Proposition 2.2] one can classify  $Rep_k(K(2))_{exc}$  and the result is:

$$(62) \quad \forall X \in Rep_k(K(2))_{exc} \quad X \cong k^{n+1} \xrightarrow[\pi_-^n]{\pi_+^n} k^n \text{ or } X \cong k^n \xrightarrow[j_-^n]{j_+^n} k^{n+1} \text{ for some } n \in \mathbb{N}.$$

Since  $\mathcal{A}$  is a bounded t-structure in  $\langle U, V \rangle$ , the inclusion functor  $\mathcal{A} \rightarrow \mathcal{T}$  induces an embedding of groups  $K(\mathcal{A}) \rightarrow K(\mathcal{T})$ . Now from (60), (61), (62) it follows that:

$$(63) \quad \{[X] \in K(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \{(n+1)[U] + n[V], \quad n[U] + (n+1)[V] : n \in \mathbb{N}\}.$$

If  $(U, V) = (E_1^{m+1}, E_1^m[-1])$ , then using (55) we obtain:

$$\begin{aligned} (n+1)[U] + n[V] &= (n+1)[E_1^{m+1}] - n[E_1^m] = (n+1)((m+1)\delta + [E_1^0]) - n(m\delta + [E_1^0]) \\ &= (n+m+1)\delta + [E_1^0] = [E_1^{n+m+1}] \\ n[U] + (n+1)[V] &= n((m+1)\delta + [E_1^0]) - (n+1)(m\delta + [E_1^0]) = (n-m)\delta - [E_1^0] \\ &= \begin{cases} [E_2^{n-m-1}] & n \geq m+1 \\ -[E_1^{m-n}] = [E_1^{m-n}[-1]] & n \leq m \end{cases}. \end{aligned}$$

Hence (63) in this case is  $\{[X] \in K(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \left\{ \begin{array}{ll} [E_1^n[-1]] & 0 \leq n \leq m \\ [E_1^n] & n \geq m+1 \\ [E_2^n] & n \in \mathbb{N} \end{array} \right\}$ . Now the

second column in the table follows easily from Lemma 3.3 and the fact that there is at most one, up to isomorphism, exceptional representation in  $Rep_k(Q)$  of a given dimension vector ([7, p. 13]).

If  $(U, V) = (E_1^0, E_2^0)$ , then using (55) and (54) we reduce (63) to  $\{[X] \in K(\mathcal{T}) : X \in \mathcal{A} \cap \mathcal{T}_{exc}\} = \{[E_1^n], [E_2^n] : n \in \mathbb{N}\}$  and the third column of the table follows.

If  $(U, V) = (E_2^m, E_2^{m+1}[-1])$ , then using (55) we obtain:

$$\begin{aligned} (n+1)[U] + n[V] &= (n+1)(m\delta + [E_2^0]) - n((m+1)\delta + [E_2^0]) = (m-n)\delta + [E_2^0] \\ &= \begin{cases} [E_2^{m-n}] & n \leq m \\ -[E_1^{n-m-1}] = [E_1^{n-m-1}[-1]] & n \geq m+1 \end{cases} \\ n[U] + (n+1)[V] &= n(m\delta + [E_2^0]) - (n+1)((m+1)\delta + [E_2^0]) \\ &= -(n+m+1)\delta - [E_2^0] = [E_2^{n+m+1}[-1]] \end{aligned}$$

and now (63) and similar arguments as in the first case give the fourth column of the table.

The case when the subscript is everywhere the second (i. e. the pairs in (59)) is obtained by substituting  $E_1$  with  $E_4$ ,  $E_2$  with  $E_3$ , and using (56) instead of (55).  $\square$

Some 1-Kronecker pairs in  $D^b(Q)$  are (see [9, table (4)]):

$$(64) \quad (M', E_1^m[-1]), (M', E_2^m), (M, E_3^m), (M, E_4^m[-1]).$$

In the following lemma are listed several short exact sequences in  $Rep_k(Q)$ . On one hand, these sequences determine the set  $\mathcal{A} \cap \mathcal{T}_{exc}$ , where  $\mathcal{A}$  is the extension closure of some of the 1-Kronecker pairs in (64), so they will be helpful when we apply Propositions 2.2 and 2.9. On the other hand, they (and their combinations) will play the role of the triangles (48) and (52) when we apply Lemmas 2.11 and 2.12.

**Lemma 3.5.** *There exist arrows in  $Rep_k(Q)$  as shown below, so that the resulting sequences are exact ( $m \in \mathbb{N}$ ):*

$$(65) \quad 0 \longrightarrow E_3^m \longrightarrow E_2^m \longrightarrow M \longrightarrow 0$$

$$(66) \quad 0 \longrightarrow M \longrightarrow E_4^m \longrightarrow E_1^m \longrightarrow 0$$

$$(67) \quad 0 \longrightarrow M' \longrightarrow E_1^{m+1} \longrightarrow E_4^m \longrightarrow 0$$

$$(68) \quad 0 \longrightarrow E_2^m \longrightarrow E_3^{m+1} \longrightarrow M' \longrightarrow 0$$

$$(69) \quad 0 \longrightarrow E_3^0 \longrightarrow M' \longrightarrow E_1^0 \longrightarrow 0.$$

*Proof.* The proof is an exercise using Proposition 3.1 . □

### 3.2. Recollection of some results of [9] with new notations.

It is useful to introduce some notations (see Proposition 3.1 for the notations  $E_i^j$ ,  $M$ ,  $M'$ ):

$$(70) \quad a^m = \begin{cases} E_1^{-m} & m \leq 0 \\ E_2^{m-1}[1] & m \geq 1 \end{cases}; \quad b^m = \begin{cases} E_4^{-m} & m \leq 0 \\ E_3^{m-1}[1] & m \geq 1 \end{cases}.$$

**Remark 3.6.** *The objects in  $\mathcal{T}_{exc}$  up to isomorphism are  $\{a^j[k], b^j[k], M[k], M'[k] : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ .*

Using [9, table (4) in Proposition 2.4]), one verifies that:

**Corollary 3.7.** *(of [9, Proposition 2.4]) For each  $m \in \mathbb{Z}$  we have:*

$$\begin{aligned} (71) \quad & \text{hom}(M', a^m) \neq 0; \quad \text{hom}(M, b^m) \neq 0; \quad \text{hom}^*(a^m, M') = 0; \\ (72) \quad & \text{hom}^1(a^m, M) \neq 0; \quad \text{hom}^1(b^m, M') \neq 0; \quad \text{hom}^*(b^m, M) = 0 \\ (73) \quad & \text{hom}^1(b^{m+1}, a^n) \neq 0 \text{ for } m > n; \quad \text{hom}(b^m, a^n) \neq 0 \text{ for } m \leq n; \quad \text{hom}^*(b^{m+1}, a^m) = 0 \\ (74) \quad & \text{hom}^1(a^m, b^n) \neq 0 \text{ for } m > n; \quad \text{hom}(a^m, b^{n+1}) \neq 0 \text{ for } m \leq n; \quad \text{hom}^*(a^m, b^m) = 0; \\ (75) \quad & \text{hom}(a^m, a^n) \neq 0 \text{ for } m \leq n; \quad \text{hom}^1(a^m, a^n) \neq 0 \text{ for } m > n + 1; \quad \text{hom}^*(a^m, a^{m-1}) = 0 \\ (76) \quad & \text{hom}(b^m, b^n) \neq 0 \text{ for } m \leq n; \quad \text{hom}^1(b^m, b^n) \neq 0 \text{ for } m > n + 1; \quad \text{hom}^*(b^m, b^{m-1}) = 0 \\ (77) \quad & \text{hom}^1(M, M') \neq 0 \quad \text{hom}^1(M', M) \neq 0. \end{aligned}$$

It is useful to keep in mind the following remarks:

**Remark 3.8.** *Recall that  $\phi_-(A) > \phi_+(B)$  implies  $\text{hom}(A, B) = 0$  and in particular  $\text{hom}(A, B) \neq 0$  implies  $\phi_-(A) \leq \phi_+(B)$  (for each stability condition).*

*Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either  $\{a^i\}_{i \in \mathbb{Z}}$  or  $\{b^i\}_{i \in \mathbb{Z}}$ . From (75) and (76) we see that:*

(a) *For  $m \leq n$  we have  $\text{hom}(x^m, x^n) \neq 0$ . In particular, if  $x^m, x^n \in \sigma^{ss}$  and  $m \leq n$  then  $\phi(x^m) \leq \phi(x^n)$ .*

(b) *For  $m + 1 < n$  we have  $\text{hom}^1(x^n, x^m) \neq 0$ . In particular, if  $x^m, x^n \in \sigma^{ss}$  and  $m + 1 < n$  then  $\phi(x^n) \leq \phi(x^m) + 1$ .*

**Remark 3.9.** *Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either  $\{a^i\}_{i \in \mathbb{Z}}$  or  $\{b^i\}_{i \in \mathbb{Z}}$ . Lemma 3.4 in terms of the notations (70) is equivalent to saying that for any three integers  $i \leq p$ ,  $p + 1 \leq j$  we have that  $x^i$  and  $x^j[-1]$  are in the extension closure of  $\{x^p, x^{p+1}[-1]\}$ .*

Keeping these remarks in mind one easily proves:

**Lemma 3.10.** *Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either  $\{a^i\}_{i \in \mathbb{Z}}$  or  $\{b^i\}_{i \in \mathbb{Z}}$ . If there exists  $m \in \mathbb{Z}$  such that  $\{x^m, x^{m+1}\} \subset \sigma^{ss}$  and  $\phi(x^m) + 1 < \phi(x^{m+1})$ , then for  $i \notin \{m, m + 1\}$  we have  $x^i \notin \sigma^{ss}$ .*

*Proof.* Suppose  $x^i \in \sigma^{ss}$  with  $i < m$ , then by Remark (a) we have  $3.8 \phi(x^i) + 1 \leq \phi(x^m) + 1 < \phi(x^{m+1})$ , hence  $\text{hom}^1(x^{m+1}, x^i) = 0$ , which contradicts the second part of Remark 3.8 (b). If  $x^i \in \sigma^{ss}$  with  $i > m + 1$ , then by Remark 3.8 (a) we obtain  $\text{hom}^1(x^i, x^m) = 0$ , which again contradicts Remark 3.8 (b). □

We will use often the following result obtained in [9]:

**Corollary 3.11.** [9, Corollary 2.6 (b)] *For any two exceptional objects  $X, Y \in D^b(Q)$  at most one element of the family  $\{\mathrm{hom}^p(X, Y)\}_{p \in \mathbb{Z}}$  is nonzero.*

Due to Corollary 3.11 we can apply Lemmas 2.11, 2.12 to  $D^b(Q)$ . Furthermore, we have:

**Corollary 3.12.** ([9]) *The full exceptional collections in  $D^b(Q)$  up to isomorphism and shifts are in the set of triples  $\mathfrak{T}$  given below. Propositions 2.7, 2.9, 2.10 can be applied to any of these triples.*

$$\mathfrak{T} = \left\{ \begin{array}{ccc} (M', a^m, a^{m+1}) & (a^m, b^{m+1}, a^{m+1}) & (a^m, a^{m+1}, M) \\ (M, b^m, b^{m+1}) & (b^m, a^m, b^{m+1}) & (b^m, b^{m+1}, M') \\ (b^m, M', a^m) & (a^m, M, b^{m+1}) & . \end{array} : m \in \mathbb{N} \right\}.$$

*Proof.* The list  $\mathfrak{T}$  follows straightforwardly from [9, Corollary 2.9]. By Corollary 3.7  $\mathrm{hom}^*(X, Y) \neq 0$  for any exceptional pair  $(X, Y)$ , therefore Propositions 2.7, 2.9 can be applied to any of the triples. Proposition 2.10 can be applied due to Corollary 3.11.  $\square$

**Remark 3.13.** *It is known [6] that the Braid group on three strings  $B_3$  acts transitively on the exceptional triples of  $\mathrm{Rep}_k(Q)$ . This action is not free (see [9, Remark 2.12]).*

**Remark 3.14.** *With the notations (70) the two orbits of 2-Kronecker pairs (see (58) and (59)) are  $\{(a^m, a^{m+1}[-1])\}_{m \in \mathbb{Z}}$  and  $\{(b^m, b^{m+1}[-1])\}_{m \in \mathbb{Z}}$ . Each of these pairs can be extended to three non-equivalent triples, so we obtain two sets of triples. Having the list  $\mathfrak{T}$  above, we see that these two sets of triples are:*

$$(78) \quad \mathfrak{T}_a = \{(M', a^m, a^{m+1}), (a^m, b^{m+1}, a^{m+1}), (a^m, a^{m+1}, M) : m \in \mathbb{Z}\}$$

$$(79) \quad \mathfrak{T}_b = \{(M, b^m, b^{m+1}), (b^m, a^m, b^{m+1}), (b^m, b^{m+1}, M') : m \in \mathbb{Z}\}.$$

Furthermore we have:

$$(80) \quad \mathfrak{T} = \mathfrak{T}_a \cup \{(b^m, M', a^m), (a^m, M, b^{m+1}) : m \in \mathbb{Z}\} \cup \mathfrak{T}_b \quad \mathfrak{T}_a \cap \mathfrak{T}_b = \emptyset.$$

**Remark 3.15.** *The short exact sequences (67), (68), (69) in terms of the notations (70) become a sequence of distinguished triangles (for each  $p$ ):*

$$(81) \quad \begin{array}{ccc} b^{p+1}[-1] & \xrightarrow{\quad} & M' \\ & \searrow \text{dashed} & \swarrow \\ & a^p & \end{array}$$

The short exact sequences (65) and (66) become the following distinguished triangles ( $q \in \mathbb{Z}$ ):

$$(82) \quad \begin{array}{ccc} a^q[-1] & \xrightarrow{\quad} & M \\ & \searrow \text{dashed} & \swarrow \\ & b^q & \end{array}.$$

### 3.3. Comments on the vectors $\{Z(X) : X \in \mathcal{T}_{exc}\}$ .

Corollary 3.2 shows that for each  $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{T})$  we have  $Z(\delta) = Z(E_1^0) + Z(E_2^0) = Z(E_4^0) + Z(E_3^0)$  and  $Z(E_k^m) = mZ(\delta) + Z(E_k^0)$  for each  $m \in \mathbb{N}$  and each  $k = 1, 2, 3, 4$ . Due to these equalities, with the notations (70) we can write (recall that  $Z(X[j]) = (-1)^j Z(X)$  for any  $j \in \mathbb{Z}$ ,  $X \in \mathcal{T}$ ):

$$(83) \quad \forall j \in \mathbb{Z} \quad Z(a^{j+1}) = Z(a^j) - Z(\delta) \quad \text{and} \quad Z(b^{j+1}) = Z(b^j) - Z(\delta).$$

Therefore for any two integers  $m, n$  we have:

$$(84) \quad Z(a^m) = Z(a^n) - (m - n)Z(\delta) \quad \text{and} \quad Z(b^m) = Z(b^n) - (m - n)Z(\delta).$$

Next we discuss collinear vectors among  $\{Z(a^j)\}_{j \in \mathbb{Z}}$  and  $\{Z(b^j)\}_{j \in \mathbb{Z}}$ . We fix first the meaning of “collinear”:

**Definition 3.16.** We say that a family  $\{A_i\}_{i \in I}$  of complex numbers is collinear if  $\{A_i\}_{i \in I} \subset \mathbb{R}c$  for some  $c \in \mathbb{C} \setminus \{0\}$ . In particular,  $0 \in \mathbb{C}$  is collinear to any  $a \in \mathbb{C}$ .

With this definition we have:

**Lemma 3.17.** Let  $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$ . Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either  $\{a^i\}_{i \in \mathbb{Z}}$  or  $\{b^i\}_{i \in \mathbb{Z}}$ . Recall that  $\delta$  is defined in (54) and consider a sequence in  $\mathbb{C}$  (infinite in both directions) of the form:

$$(85) \quad \dots, Z(x^{-i}), \dots, Z(x^{-2}), Z(x^{-1}), Z(x^0), Z(\delta), Z(x^1), Z(x^2), Z(x^3), \dots, Z(x^j), \dots$$

Then the following conditions are equivalent: (a) Two of the vectors in this sequence are collinear; (b) The entire sequence is collinear.

*Proof.* Recall that formula (84) holds for any  $m, n \in \mathbb{Z}$ .

If  $Z(x^i)$  and  $Z(\delta)$  are collinear for some  $i \in \mathbb{Z}$ , then  $Z(\delta)$  and  $Z(x^0)$  are collinear by  $Z(x^0) = Z(x^i) + iZ(\delta)$  and (b) follows from the equalities  $Z(x^j) = Z(x^0) - jZ(\delta)$ ,  $j \in \mathbb{Z}$ .

If  $Z(x^i)$  and  $Z(x^j)$  are collinear for some  $i \neq j$ , then by the equality  $Z(\delta) = \frac{1}{j-i}(Z(x^i) - Z(x^j))$  we see that  $Z(\delta)$  and  $Z(x^i)$  are collinear and (b) follows from the considered case above  $\square$

**Corollary 3.18.** Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either  $\{a^i\}_{i \in \mathbb{Z}}$  or  $\{b^i\}_{i \in \mathbb{Z}}$ . Let two of the vectors in the sequence (85) be non-collinear. Then:

- (a) All the vectors in this sequence are non-zero and no two of them are collinear.
- (b) If for two integers  $n \neq m$  holds  $\{x^n, x^m\} \subset \sigma^{ss}$ , then we have  $\phi(x^n) \notin \phi(x^m) + \mathbb{Z}$ .
- (c) The numbers  $\{Z(x^j)\}_{j \in \mathbb{Z}}$  are contained in a common connected component of  $\mathbb{C} \setminus \mathbb{R}Z(\delta)$ .
- (d) If for two integers  $n < m$  we have  $\{x^n, x^m\} \subset \sigma^{ss}$  and  $\phi(x^m) < \phi(x^n) + 1$ , then:<sup>7</sup>

$$\{Z(x^j)\}_{j \in \mathbb{N}} \subset Z(\delta)_+^c.$$

*Proof.* (a) and (b) follow from Lemma 3.17, and the following axiom in [1]:

$$(86) \quad X \in \sigma^{ss} \quad \Rightarrow \quad Z(X) = r(X) \exp(i\pi\phi(X)), \quad r(X) > 0.$$

Since  $Z(x^0)$  and  $Z(\delta)$  are non-collinear, it follows that either  $Z(x^0) \in Z(\delta)_+^c$  or  $Z(x^0) \in Z(\delta)_-^c$ . From formula (84) we have  $Z(x^j) = Z(x^0) - jZ(\delta)$  for any  $j \in \mathbb{Z}$  therefore either  $\{Z(x^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$  or  $\{Z(x^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_-^c$ . Therefore we obtain (c). Now to show (d), it is enough to show that  $Z(x^m) \in Z(\delta)_+^c$ . From (b) and Remark 3.8 (a) we get the inequalities  $\phi(x^n) < \phi(x^m) < \phi(x^n) + 1$ . By drawing a picture and taking into account formula (86) and the equality  $Z(x^n) = Z(x^m) + (m - n)Z(\delta)$ , one sees that  $\phi(x^n) < \phi(x^m) < \phi(x^n) + 1$  is impossible if  $Z(x^m) \in Z(\delta)_-^c$ .  $\square$

**Corollary 3.19.** Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either the sequence  $\{a^i\}_{i \in \mathbb{Z}}$  or the sequence  $\{b^i\}_{i \in \mathbb{Z}}$ .

If  $Z(\delta) \neq 0$  and  $Z(x^q) \in Z(\delta)_+^c$  for some  $q \in \mathbb{Z}$ , then  $\{Z(x^i)\}_{i \in \mathbb{Z}} \subset Z(\delta)_+^c$  and for any  $t \in \mathbb{R}$  with  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$  we have:

$$(87) \quad \forall p \in \mathbb{Z} \quad \arg_{(t, t+1)}(Z(x^p)) < \arg_{(t, t+1)}(Z(x^{p+1}))$$

$$(88) \quad \lim_{p \rightarrow -\infty} \arg_{(t, t+1)}(Z(x^p)) = t; \quad \lim_{p \rightarrow +\infty} \arg_{(t, t+1)}(Z(x^p)) = t + 1.$$

*Proof.* Since  $Z(\delta)$  and  $Z(x^q)$  are not collinear by Corollary 3.18 (c) and  $Z(x^q) \in Z(\delta)_+^c$  it follows that  $\{Z(x^i)\}_{i \in \mathbb{Z}} \subset Z(\delta)_+^c$ . The inequalities (87) follow from  $Z(x^{p+1}), Z(x^p) \in Z(\delta)_+^c$  and  $Z(x^{p+1}) = Z(x^p) - Z(\delta)$  (see (84)). The formulas in (88) follow also from (84) and  $\{Z(x^i)\}_{i \in \mathbb{Z}} \subset Z(\delta)_+^c$ .  $\square$

<sup>7</sup>See (2) for the notations  $Z(\delta)_\pm^c$ .

**Corollary 3.20.** *Let  $Z(M)$  and  $Z(M')$  be non-zero and have the same direction.<sup>8</sup> Let  $Z(a^q), Z(b^p) \in Z(\delta)_+^c$  for some  $p, q \in \mathbb{Z}$ .*

*Then  $\{Z(a^j), Z(b^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$  and for any  $t \in \mathbb{R}$  with  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$  the formulas (88), (87) hold for both the sequences  $\{Z(a^j)\}_{j \in \mathbb{Z}}$  and  $\{Z(b^j)\}_{j \in \mathbb{Z}}$ .*

*Furthermore, for any three integers  $i, j, m$  we have:*

$$(89) \quad j < m \leq i \Rightarrow \arg_{(t, t+1)}(Z(a^j)) < \arg_{(t, t+1)}(Z(b^m)) < \arg_{(t, t+1)}(Z(a^i)).$$

*Proof.* Corollary 3.19 shows the first part of the conclusion. To show (89) we note first that the equalities (57) with the notations (70) become the following (for any  $m \in \mathbb{Z}$ ):

$$(90) \quad Z(b^m) - Z(M) = Z(a^m) \quad Z(b^m) + Z(M') = Z(a^{m-1}).$$

Since  $Z(a^{m-1}), Z(a^m), Z(b^m) \in Z(\delta)_+^c$  for any  $m \in \mathbb{Z}$  and  $Z(M), Z(M')$  have the same direction as  $Z(\delta)$  (recall (54)) the equalities (90) imply that  $\arg_{(t, t+1)}(Z(a^{m-1})) < \arg_{(t, t+1)}(Z(b^m)) < \arg_{(t, t+1)}(Z(a^m))$  for any  $m \in \mathbb{Z}$ . Now (89) follows from (87) (applied to the case  $\{x^i\}_{i \in \mathbb{Z}} = \{a^i\}_{i \in \mathbb{Z}}$ ).  $\square$

#### 4. THE UNION $\text{Stab}(D^b(Q)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}$

In this Section we distinguish some building blocks of  $\text{Stab}(D^b(Q))$  and organize them in a manner consistent with the order in which we will glue these blocks in the next sections.

Theorem 1.1 says that for each  $\sigma \in \text{Stab}(D^b(Q))$  there exists a  $\sigma$ -triple. This means that (see [9, Corollary 3.20]) for each  $\sigma$  there exists an Ext-exceptional triple  $\mathcal{E}$  with  $\sigma \in \Theta'_\mathcal{E}$ . From Corollary 3.12 we see that  $\mathcal{E}$  is a shift of some of the triples in  $\mathfrak{T}$ . Recalling the notation (9) we get  $\text{Stab}(D^b(Q)) = \bigcup_{\mathcal{E} \in \mathfrak{T}} \Theta_\mathcal{E}$ . Our basic building blocks are  $\{\Theta_\mathcal{E}\}_{\mathcal{E} \in \mathfrak{T}}$  and by Proposition 2.7 they are contractible.

For a given triple  $(A, B, C) \in \mathfrak{T}$  we will denote the open subset  $\Theta_{(A, B, C)} \subset \text{Stab}(D^b(Q))$  by  $(A, B, C)$ , when (we believe that) no confusion may arise. With this convention we can write

$$(91) \quad \text{Stab}(D^b(Q)) = \bigcup_{(A, B, C) \in \mathfrak{T}} (A, B, C).$$

For a given  $X \in \{M, M'\}$  we denote by  $(X, \_, \_)$  the following open subset of  $\text{Stab}(D^b(Q))$ :

$$(92) \quad \text{Stab}(D^b(Q)) \supset (X, \_, \_) = \bigcup_{\{(B_0, B_1, B_2) \in \mathfrak{T} : B_0 = X\}} (X, B_1, B_2).$$

Similarly we define  $(\_, X, \_)$  and  $(\_, \_, X)$ . Looking at the list  $\mathfrak{T}$  and denoting (see (78), (79)):

$$(93) \quad \mathfrak{T}_a^{st} = \bigcup_{(A, B, C) \in \mathfrak{T}_a} (A, B, C) \subset \text{Stab}(D^b(Q)); \quad \mathfrak{T}_b^{st} = \bigcup_{(A, B, C) \in \mathfrak{T}_b} (A, B, C) \subset \text{Stab}(D^b(Q))$$

we can regroup the union (91) using (80) as follows:

$$(94) \quad \text{Stab}(D^b(Q)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup (\_, M', \_) \cup \mathfrak{T}_b^{st}.$$

---

<sup>8</sup>We mean that  $Z(M) = yZ(M')$  for some  $y \in \mathbb{R}_{>0}$ . In particular, by  $Z(\delta) = Z(M) + Z(M')$  (recall (54)) we see that  $Z(\delta)$  is non-zero.

**Remark 4.1.** From the very definition (9) of  $\Theta_{\mathcal{E}}$  it is clear that  $\Theta_{\mathcal{E}[\mathbf{p}]} = \Theta_{\mathcal{E}}$  for any triple  $\mathcal{E} = (A, B, C) \in \mathfrak{T}$  and any  $\mathbf{p} \in \mathbb{Z}^3$ . Using the notations explained here, we have  $(A, B, C) = (A[p_0], B[p_1], C[p_2]) \subset \text{Stab}(D^b(Q))$  for any  $p_0, p_1, p_2 \in \mathbb{Z}$ .

#### 5. SOME CONTRACTIBLE SUBSETS OF $\mathfrak{T}_a^{st}$ AND $\mathfrak{T}_b^{st}$ . PROOF THAT $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$

In this Section is shown that  $(X, \_, \_)$  and  $(\_, \_, X)$  are contractible subsets of  $\text{Stab}(\mathcal{T})$  for any  $X \in \{M, M'\}$  and that  $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$ .

We will refer often to some of the formulas in Corollary 3.7. Whenever we discuss  $\text{hom}(A, B)$  or  $\text{hom}^1(A, B)$  with  $A, B$  varying in the symbols  $M, M', a^m, b^m, m \in \mathbb{Z}$ , we refer to Corollary 3.7.

Putting (78), (79) in (93) we obtain:

$$(95) \quad \mathfrak{T}_a^{st} = (M', \_, \_) \cup (\_, \_, M) \cup \bigcup_{p \in \mathbb{Z}} (a^p, b^{p+1}, a^{p+1})$$

$$(96) \quad \mathfrak{T}_b^{st} = (M, \_, \_) \cup (\_, \_, M') \cup \bigcup_{q \in \mathbb{Z}} (b^q, a^q, b^{q+1})$$

$$(97) \quad (M', \_, \_) = \bigcup_{m \in \mathbb{Z}} (M', a^m, a^{m+1}); \quad (M, \_, \_) = \bigcup_{m \in \mathbb{Z}} (M, b^m, b^{m+1})$$

$$(98) \quad (\_, \_, M) = \bigcup_{m \in \mathbb{Z}} (a^m, a^{m+1}, M); \quad (\_, \_, M') = \bigcup_{m \in \mathbb{Z}} (b^m, b^{m+1}, M').$$

We apply Proposition 2.7 to the triples  $(a^p, b^{p+1}, a^{p+1})$  and  $(b^q, a^q, b^{q+1})$ . Using Corollary 3.11 and the formulas in Corollary 3.7 we see that in both the cases the coefficients  $\alpha, \beta, \gamma$  defined in (29) are  $\alpha = \beta = \gamma = -1$ . Thus, we obtain the following formulas for the sets  $(a^p, b^{p+1}, a^{p+1}) \subset \text{Stab}(D^b(\mathcal{T}))$  and  $(b^q, a^q, b^{q+1}) \subset \text{Stab}(D^b(\mathcal{T}))$  in the first and the second column, respectively:

$$(99) \quad \begin{array}{|c|c|} \hline (a^p, b^{p+1}, a^{p+1}) & (b^q, a^q, b^{q+1}) \\ \hline \left\{ \begin{array}{l} \phi(a^p) < \phi(b^{p+1}) \\ a^p, b^{p+1}, a^{p+1} \in \sigma^{ss} : \phi(a^p) + 1 < \phi(a^{p+1}) \\ \phi(b^{p+1}) < \phi(a^{p+1}) \end{array} \right\} & \left\{ \begin{array}{l} \phi(b^q) < \phi(a^q) \\ b^q, a^q, b^{q+1} \in \sigma^{ss} : \phi(b^q) + 1 < \phi(b^{q+1}) \\ \phi(a^q) < \phi(b^{q+1}) \end{array} \right\} \\ \hline \end{array}$$

Similarly, applying Proposition 2.7 to the triples in the unions (98), (97) (with the help of Corollary 3.7 and Corollary 3.11) we see that  $(M', \_, \_) \cup (\_, \_, M)$  and  $(M, \_, \_) \cup (\_, \_, M')$  are the unions of the sets in the first and the second column of the following table, respectively (where  $m, n, i, j$  vary throughout  $\mathbb{Z}$ ):

$$(100) \quad \begin{array}{|c|c|} \hline (M', \_, \_) \cup (\_, \_, M) & (M, \_, \_) \cup (\_, \_, M') \\ \hline \left\{ \begin{array}{l} \phi(M') < \phi(a^j) \\ M', a^j, a^{j+1} \in \sigma^{ss} : \phi(M') + 1 < \phi(a^{j+1}) \\ \phi(a^j) < \phi(a^{j+1}) \end{array} \right\} & \left\{ \begin{array}{l} \phi(M) < \phi(b^n) \\ M, b^n, b^{n+1} \in \sigma^{ss} : \phi(M) + 1 < \phi(b^{n+1}) \\ \phi(b^n) < \phi(b^{n+1}) \end{array} \right\} \\ \hline \left\{ \begin{array}{l} \phi(a^m) < \phi(a^{m+1}) \\ a^m, a^{m+1}, M \in \sigma^{ss} : \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array} \right\} & \left\{ \begin{array}{l} \phi(b^i) < \phi(b^{i+1}) \\ b^i, b^{i+1}, M' \in \sigma^{ss} : \phi(b^i) < \phi(M') \\ \phi(b^{i+1}) < \phi(M') + 1 \end{array} \right\} \\ \hline \end{array}$$



For the triples on the first row of table (100) we have  $\alpha = \beta = \gamma - 1$  and for the triples on the second row we have  $\alpha = -1, \beta = \gamma = 0$  (one shows this using Corollaries 3.7 and 3.11).

**5.1. Proof that  $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$ .**

Recall the axioms of Bridgeland [1], that  $\phi(A[1]) = \phi(A) + 1$  for any  $A \in \sigma^{ss}$ , and that  $A, B \in \sigma^{ss}$  and  $\phi(A) > \phi(B)$  imply  $\text{hom}(A, B) = 0$ . We will use these axiom often implicitly. We start with:

**Lemma 5.1.**  $((M', \_, \_) \cup (\_, \_, M)) \cap ((M, \_, \_) \cup (\_, \_, M')) = \emptyset$ .

*Proof.* Suppose  $\sigma \in (a^m, a^{m+1}, M) \cap (M, b^n, b^{n+1})$ , then by the table (100) we obtain  $\text{hom}^1(b^{n+1}, a^m) = 0$  and  $\text{hom}(b^{n+1}, a^{m+1}) = 0$ , which contradicts (73).

Suppose  $\sigma \in (a^m, a^{m+1}, M) \cap (b^i, b^{i+1}, M')$ , then by  $\text{hom}(M', a^m) \neq 0$  (see (71)) and table (100) we obtain  $\phi(b^i) < \phi(M') \leq \phi(a^m) < \phi(M)$ , which contradicts  $\text{hom}(M, b^i) \neq 0$  (see (71)).

Suppose  $\sigma \in (M', a^j, a^{j+1}) \cap (M, b^n, b^{n+1})$ , then by  $\text{hom}^1(a^{j+1}, M) \neq 0$  (see (72)) and table (100) we obtain  $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(M) + 1 < \phi(b^{n+1})$ , which contradicts  $\text{hom}^1(b^{n+1}, M') \neq 0$ .

Suppose  $\sigma \in (M', a^j, a^{j+1}) \cap (b^i, b^{i+1}, M')$ , then by the table we have  $\text{hom}^1(a^{j+1}, b^i) = 0$ ,  $\text{hom}(a^{j+1}, b^{i+1}) = 0$ , which contradicts (74). The lemma is proved.  $\square$

**Lemma 5.2.** For any  $p, q \in \mathbb{Z}$  we have  $(a^p, b^{p+1}, a^{p+1}) \cap (b^q, a^q, b^{q+1}) = \emptyset$ .

*Proof.* Let  $\sigma \in (a^p, b^{p+1}, a^{p+1})$ , then in table (99) we see that  $a^p, a^{p+1} \in \sigma^{ss}$  and  $\phi(a^p) + 1 < \phi(a^{p+1})$ . Now by Lemma 3.10 we see that  $a^q \notin \sigma^{ss}$  for  $q \notin \{p, p+1\}$ , and therefore  $\sigma \notin (b^q, a^q, b^{q+1})$  for  $q \notin \{p, p+1\}$ .

Suppose that  $\sigma \in (b^p, a^p, b^{p+1})$ , then from table (99) we obtain  $\phi(b^p) + 1 < \phi(a^p) + 1 < \phi(a^{p+1})$ , hence  $\text{hom}^1(a^{p+1}, b^p) = 0$ , which contradicts (74).

Suppose that  $\sigma \in (b^{p+1}, a^{p+1}, b^{p+2})$ , then from table (99) we obtain  $\phi(a^p) + 1 < \phi(a^{p+1}) < \phi(b^{p+2})$ , hence  $\text{hom}^1(b^{p+2}, a^p) = 0$ , which contradicts (73). The lemma is proved.  $\square$

**Lemma 5.3.** For any  $p, q \in \mathbb{Z}$  we have:  $((M', \_, \_) \cup (\_, \_, M)) \cap (b^q, a^q, b^{q+1}) = \emptyset$  and  $((M, \_, \_) \cup (\_, \_, M')) \cap (a^p, b^{p+1}, a^{p+1}) = \emptyset$ .

*Proof.* Assume first that  $\sigma \in (b^q, a^q, b^{q+1})$ , then table (99) shows that:

$$(101) \quad \phi(b^q) + 1 < \phi(b^{q+1}).$$

Suppose that  $\sigma \in (\_, \_, M)$ , then using (101),  $\text{hom}(M, b^q) \neq 0$  and table (100) we see that  $\phi(a^m) + 1 < \phi(b^{q+1})$  and  $\phi(a^{m+1}) < \phi(b^{q+1})$  for some  $m \in \mathbb{Z}$ , hence  $\text{hom}^1(b^{q+1}, a^m) = \text{hom}(b^{q+1}, a^{m+1}) = 0$ , which contradicts (73).

Suppose that  $\sigma \in (M', \_, \_)$ , then using (101),  $\text{hom}^1(b^{q+1}, M') \neq 0$  and table (100) we see that  $\phi(b^q) + 1 < \phi(a^{j+1})$  and  $\phi(b^q) < \phi(a^j)$  for some  $j \in \mathbb{Z}$ , hence  $\text{hom}^1(a^{j+1}, b^q) = \text{hom}(a^j, b^q) = 0$ , which contradicts (74). So far we proved that  $((\_, \_, M) \cup (M', \_, \_)) \cap (b^q, a^q, b^{q+1}) = \emptyset$ .

Assume now that  $\sigma \in (a^p, b^{p+1}, a^{p+1})$ , then table (99) shows that:

$$(102) \quad \phi(a^p) + 1 < \phi(a^{p+1}).$$

Suppose that  $\sigma \in (\_, \_, M')$ , then using (102),  $\text{hom}(M', a^p) \neq 0$  and table (100) we see that  $\phi(b^i) + 1 < \phi(a^{p+1})$  and  $\phi(b^{i+1}) < \phi(a^{p+1})$  for some  $i \in \mathbb{Z}$ , hence  $\text{hom}^1(a^{p+1}, b^i) = \text{hom}(a^{p+1}, b^{i+1}) = 0$ , which contradicts (74).

Suppose that  $\sigma \in (M, \_, \_)$ , then using (102),  $\text{hom}^1(a^{p+1}, M) \neq 0$  and table (100) we see that  $\phi(a^p) + 1 < \phi(b^{n+1})$  and  $\phi(a^p) < \phi(b^n)$  for some  $n \in \mathbb{Z}$ , hence  $\text{hom}^1(b^{n+1}, a^p) = \text{hom}(b^n, a^p) = 0$ , which contradicts (73). Thus, we proved the second equality as well.  $\square$

Lemmas 5.1, 5.2, 5.3, and formulas (95), (96) imply that  $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$ .

**5.2. The subsets  $(\_, \_, M)$ ,  $(\_, \_, M')$ ,  $(M, \_, \_)$  and  $(M', \_, \_)$  are contractible.**

We start with:

**Lemma 5.4.** *Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either the sequence  $\{a^i\}_{i \in \mathbb{Z}}$  or the sequence  $\{b^i\}_{i \in \mathbb{Z}}$ . If  $m > j$  then:*

$$(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X) = \left\{ \sigma : \begin{array}{ll} x^m \in \sigma^{ss} & 0 < \phi(x^{m+1}) - \phi(x^m) < 1 \\ x^{m+1} \in \sigma^{ss} & \phi(x^m) < \phi(X) \\ X \in \sigma^{ss} & \phi(x^{m+1}) < \phi(X) + 1 \end{array} \right\},$$

where  $X = M$  if  $\{x^i\}_{i \in \mathbb{Z}} = \{a^i\}_{i \in \mathbb{Z}}$ , and  $X = M'$  if  $\{x^i\}_{i \in \mathbb{Z}} = \{b^i\}_{i \in \mathbb{Z}}$ .

In particular,  $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$  and  $(x^m, x^{m+1}, X) \cup (x^j, x^{j+1}, X)$  are contractible.

*Proof.* We show first the inclusion  $\subset$ . Assume that  $\sigma \in (x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$  and  $m > j$ . Then  $X, x^{m+1}, x^m, x^{j+1}, x^j$  are all semistable and by table (100) we have

$$(103) \quad \begin{array}{ll} \phi(x^m) < \phi(x^{m+1}) & \phi(x^j) < \phi(x^{j+1}) \\ \phi(x^m) < \phi(X) & \phi(x^j) < \phi(X) \\ \phi(x^{m+1}) < \phi(X) + 1 & \phi(x^{j+1}) < \phi(X) + 1 \end{array}.$$

By  $m > j$  it follows  $\text{hom}^1(x^{m+1}, x^j) \neq 0$ , hence  $\phi(x^{m+1}) \leq \phi(x^j) + 1$  (see Remark 3.8 (b)). On the other hand from the inequalities above we have  $\phi(x^j) + 1 < \phi(x^{j+1}) + 1$  and by Remark 3.8 (a) we obtain  $\phi(x^{j+1}) + 1 \leq \phi(x^m) + 1$ . Thus we obtain  $\phi(x^{m+1}) < \phi(x^m) + 1$  and  $\subset$  follows.

We show now  $\supset$ . The condition defining the set on the right-hand side is the same as  $\sigma \in (x^m, x^{m+1}, X)$  and  $\phi(x^m) > \phi(x^{m+1}[-1])$  (see table (100)). From Proposition 2.9 (a) it follows that  $\mathcal{A} \cap \mathcal{T}_{exc} \subset \sigma^{ss}$ , where  $\mathcal{A}$  is the extension closure of  $(x^m, x^{m+1}[-1])$ , hence By Remark 3.9 we have  $\{x^{j+1}, x^j\} \subset \sigma^{ss}$ . The inequality  $0 < \phi(x^{m+1}) - \phi(x^m) < 1$  and (86) show that  $Z(x^{m+1}), Z(x^m)$  are not collinear, hence by Corollary 3.18 (b) we get  $\phi(x^{j+1}) \neq \phi(x^j)$ . Now by Remark 3.8 (a) and  $\phi(x^j) < \phi(x^{j+1})$

the incidence  $\sigma \in (x^m, x^{m+1}, X)$  we get:  $\phi(x^j) \leq \phi(x^m) < \phi(X)$ . In table (100) we see  $\phi(x^{j+1}) \leq \phi(x^{m+1}) < \phi(X) + 1$

that  $\sigma \in (x^j, x^{j+1}, X)$  and the inclusion  $\supset$  is shown.

The proved equality implies that  $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$  is contractible (see the arguments for the proof that (41) is contractible in Proposition 2.10). Since  $(x^m, x^{m+1}, X)$  and  $(x^j, x^{j+1}, X)$  are contractible, by Remark A.6 we see that  $(x^m, x^{m+1}, X) \cup (x^j, x^{j+1}, X)$  is contractible as well.  $\square$

**Corollary 5.5.** *The subsets  $(\_, \_, M)$  and  $(\_, \_, M')$  of  $\text{Stab}(D^b(Q))$  are contractible.*

*Proof.* Recalling (98) and using the notations of the previous lemma, we see that we have to show that  $\bigcup_{j \in \mathbb{Z}} (x^j, x^{j+1}, X)$  is contractible. It is shown in Lemma 5.4, that for a given  $m \in \mathbb{Z}$  the intersection  $(x^m, x^{m+1}, X) \cap (x^j, x^{j+1}, X)$  is contractible and it is the same for all  $j < m$ . Now by induction and using Remark A.6 one shows that  $\bigcup_{k=0}^n (x^{m-k}, x^{m-k+1}, X)$  is contractible for any  $n \in \mathbb{N}$  and any  $m \in \mathbb{Z}$ . Using again Remark A.6 we deduce that  $\bigcup_{j \in \mathbb{Z}} (x^j, x^{j+1}, X)$  is contractible. The corollary follows.  $\square$

The steps in the proof that  $(M, \_, \_)$  and  $(M', \_, \_)$  are analogous. We show first:

**Lemma 5.6.** *Let  $\{x^i\}_{i \in \mathbb{Z}}$  be either the sequence  $\{a^i\}_{i \in \mathbb{Z}}$  or the sequence  $\{b^i\}_{i \in \mathbb{Z}}$ . If  $m < j$ , then:*

$$(104) \quad (X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1}) = \left\{ \sigma : \begin{array}{ll} X \in \sigma^{ss} & \phi(X) < \phi(x^m) \\ x^m \in \sigma^{ss} & \phi(X) + 1 < \phi(x^{m+1}) \\ x^{m+1} \in \sigma^{ss} & 0 < \phi(x^{m+1}) - \phi(x^m) < 1 \end{array} \right\}$$

where  $X = M'$  if  $\{x^i\}_{i \in \mathbb{Z}} = \{a^i\}_{i \in \mathbb{Z}}$ , and  $X = M$  if  $\{x^i\}_{i \in \mathbb{Z}} = \{b^i\}_{i \in \mathbb{Z}}$ .

In particular,  $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$  and  $(X, x^m, x^{m+1}) \cup (X, x^j, x^{j+1})$  are contractible.

*Proof.* By table (100) we see that the condition defining the set on the right-hand side of (104) is the same as  $\sigma \in (X, x^m, x^{m+1})$  and  $\phi(x^m) > \phi(x^{m+1}[-1])$ .

The inclusion  $\subset$  follows from table (100),  $\text{hom}^1(x^{j+1}, x^m) \neq 0$  and Remark 3.8 (a) as follows  $\phi(x^{m+1}) \leq \phi(x^j) < \phi(x^{j+1}) \leq \phi(x^m) + 1$ .

To show the converse inclusion  $\supset$  in (104), assume that  $\sigma \in (X, x^m, x^{m+1})$  and  $\phi(x^m) > \phi(x^{m+1}[-1])$ . From Proposition 2.9 (b) and Remark 3.9 it follows that  $x^j, x^{j+1} \in \sigma^{ss}$ . Since we have  $0 < \phi(x^{m+1}) - \phi(x^m) < 1$ , it follows that  $Z(x^m), Z(x^{m+1})$  are not collinear, therefore by Corollary 3.18 (b) and Remark 3.8 (a) we obtain  $\phi(x^j) < \phi(x^{j+1})$ . Since  $j > m$ , by Remark 3.8 (a) we obtain also  $\phi(X) < \phi(x^m) \leq \phi(x^j)$ ,  $\phi(X) + 1 < \phi(x^{m+1}) \leq \phi(x^{j+1})$ , hence  $\sigma \in (X, x^j, x^{j+1})$ .

The proved equality implies that  $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$  is contractible (see the arguments for the proof that (41) is contractible in Proposition 2.10). Since  $(X, x^m, x^{m+1})$  and  $(X, x^j, x^{j+1})$  are contractible, by Remark A.6 we see that  $(X, x^m, x^{m+1}) \cup (X, x^j, x^{j+1})$  is contractible as well.  $\square$

**Corollary 5.7.** *The subsets  $(M, \_, \_), (M', \_, \_) \subset \text{Stab}(D^b(Q))$  are contractible.*

*Proof.* Recalling (97) and using the notations of the previous lemma, we see that we have to show that  $\bigcup_{j \in \mathbb{Z}} (X, x^j, x^{j+1})$  is contractible. From Lemma 5.6 we know that for a given  $m \in \mathbb{Z}$  the intersection  $(X, x^m, x^{m+1}) \cap (X, x^j, x^{j+1})$  is contractible and it is the same for all  $j > m$ . Now by induction and using Remark A.6 one shows that  $\bigcup_{k=0}^n (X, x^{m+k}, x^{m+k+1})$  is contractible for any  $n \in \mathbb{N}$  and any  $m \in \mathbb{Z}$ . Using again Remark A.6 we deduce that  $\bigcup_{j \in \mathbb{Z}} (X, x^j, x^{j+1})$  is contractible. The corollary follows.  $\square$

## 6. THE SUBSETS $\mathfrak{T}_a^{st}$ AND $\mathfrak{T}_b^{st}$ ARE CONTRACTIBLE

We start by showing some empty intersections:

**Lemma 6.1.** *The unions  $\bigcup_{p \in \mathbb{Z}} (a^p, b^{p+1}, a^{p+1})$  and  $\bigcup_{p \in \mathbb{Z}} (b^p, a^p, b^{p+1})$  are disjoint. Furthermore, we have:*

$$(105) \quad p \neq q \Rightarrow (a^p, b^{p+1}, a^{p+1}) \cap (a^q, a^{q+1}, M) = (a^p, b^{p+1}, a^{p+1}) \cap (M', a^q, a^{q+1}) = \emptyset$$

$$(106) \quad p \neq q \Rightarrow (b^p, a^p, b^{p+1}) \cap (b^q, b^{q+1}, M') = (b^p, a^p, b^{p+1}) \cap (M, b^q, b^{q+1}) = \emptyset.$$

*Proof.* If  $\sigma \in (a^p, b^{p+1}, a^{p+1})$ , then these exceptional objects are semistable and by table (99) we have  $\phi(a^p) + 1 < \phi(a^{p+1})$ . Now by Lemma 3.10 we see that  $a^j$  with  $j \notin \{p, p+1\}$  can not be semistable, therefore  $\sigma \notin (a^q, b^{q+1}, a^{q+1})$ ,  $\sigma \notin (a^q, a^{q+1}, M)$ , and  $\sigma \notin (M', a^q, a^{q+1})$  for  $q \neq p$ .

If  $\sigma \in (b^p, a^p, b^{p+1})$ , then  $b^p, a^p, b^{p+1}$  are semistable and by table (99) we have  $\phi(b^p) + 1 < \phi(b^{p+1})$ . Now by Lemma 3.10 we see that  $b^j$  with  $j \notin \{p, p+1\}$  can not be semistable, therefore  $\sigma \notin (b^q, a^q, b^{q+1})$ ,  $\sigma \notin (b^q, b^{q+1}, M')$ , and  $\sigma \notin (M, b^q, b^{q+1})$  for  $q \neq p$ .  $\square$

Now we attach the pairwise non-intersecting contractible blocks  $\{(a^p, b^{p+1}, a^{p+1})\}_{p \in \mathbb{Z}}$  to  $(\_, \_, M)$  and to  $(M', \_, \_)$

**Lemma 6.2.** *For any  $p \in \mathbb{Z}$  the sets  $(a^p, b^{p+1}, a^{p+1}) \cap (\_, \_, M)$ ;  $(a^p, b^{p+1}, a^{p+1}) \cap (M', \_, \_)$ ;  $(b^p, a^p, b^{p+1}) \cap (\_, \_, M')$ ; and  $(b^p, a^p, b^{p+1}) \cap (M, \_, \_)$  are non-empty and contractible.*

*Proof.* From (97), (98) and (105) it follows that:  $(a^p, b^{p+1}, a^{p+1}) \cap (\_, \_, M) = (a^p, b^{p+1}, a^{p+1}) \cap (a^p, a^{p+1}, M)$  and  $(a^p, b^{p+1}, a^{p+1}) \cap (M', \_, \_) = (a^p, b^{p+1}, a^{p+1}) \cap (M', a^p, a^{p+1})$ .

From (97), (98) and (106) it follows that:  $(b^p, a^p, b^{p+1}) \cap (\_, \_, M') = (b^p, a^p, b^{p+1}) \cap (b^p, b^{p+1}, M')$  and  $(b^p, a^p, b^{p+1}) \cap (M, \_, \_) = (b^p, a^p, b^{p+1}) \cap (M, b^p, b^{p+1})$ .

From Proposition 2.10 it follows that  $(a^p, b^{p+1}, a^{p+1}) \cap (a^p, a^{p+1}, M)$ ,  $(a^p, b^{p+1}, a^{p+1}) \cap (M', a^p, a^{p+1})$ ,  $(b^p, a^p, b^{p+1}) \cap (b^p, b^{p+1}, M')$ , and  $(b^p, a^p, b^{p+1}) \cap (M, b^p, b^{p+1})$  are contractible.

The lemma follows.  $\square$

Let us denote:

$$(107) \quad Z = (M', \_, \_) \cup \bigcup_{p \in \mathbb{Z}} (a^p, b^{p+1}, a^{p+1}).$$

Corollary 5.7 and Lemmas 6.1, 6.2 imply (recall Remark A.6) that  $Z$  is contractible. From (95) and (98) we see that:

$$(108) \quad \mathfrak{T}_a^{st} = Z \cup (\_, \_, M) = Z \cup \bigcup_{m \in \mathbb{Z}} (a^m, a^{m+1}, M).$$

We start to glue the contractible summands in formula (108). The first step is:

**Lemma 6.3.** *The set  $(a^m, a^{m+1}, M) \cap Z$  consists of the stability conditions  $\sigma$  for which  $a^m, a^{m+1}, M$  are semistable and:*

$$(109) \quad \begin{array}{l} \phi(a^m) - 1 < \phi(a^{m+1}[-1]) < \phi(a^m) \\ \phi(a^m) - 1 < \phi(M[-1]) < \phi(a^m) \\ \arg_{(\phi(a^m)-1, \phi(a^m))}(Z(a^m) - Z(a^{m+1})) > \phi(M) - 1 \end{array} \quad \text{or} \quad \begin{array}{l} \phi(M) < \phi(a^{m+1}) \\ \phi(a^m) < \phi(a^{m+1}) \\ \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array}.$$

*It follows that  $(a^m, a^{m+1}, M) \cap Z$  and  $(a^m, a^{m+1}, M) \cup Z$  are contractible.*

*Proof.* In (105) we have that  $(a^m, a^{m+1}, M) \cap (a^j, b^{j+1}, a^{j+1}) = \emptyset$  for  $j \neq m$ . Therefore (recall (107))

$$(110) \quad (a^m, a^{m+1}, M) \cap Z = (a^m, a^{m+1}, M) \cap ((a^m, b^{m+1}, a^{m+1}) \cup (M', \_, \_)).$$

We show first the inclusion  $\subset$ . Assume that  $\sigma \in (a^m, a^{m+1}, M) \cap Z$ . Then  $a^m, a^{m+1}, M$  are semistable and from table (100) we see that

$$(111) \quad \begin{array}{l} \phi(a^m) < \phi(a^{m+1}) \\ \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array}.$$

Taking into account (110) we consider two cases.

If  $\sigma \in (a^m, b^{m+1}, a^{m+1})$ , then  $b^{m+1} \in \sigma^{ss}$  and in table (99) we see that  $\phi(b^{m+1}) < \phi(a^{m+1})$ . From  $\text{hom}(M, b^{m+1}) \neq 0$  (see (71)) it follows that  $\phi(M) \leq \phi(b^{m+1}) < \phi(a^{m+1})$  and we obtain the second system of inequalities in (109).

If  $\sigma \in (M', a^j, a^{j+1})$ , then  $M', a^j, a^{j+1} \in \sigma^{ss}$  and in table (100) we see that  $\phi(M') + 1 < \phi(a^{j+1})$  and  $\phi(a^j) < \phi(a^{j+1})$ . From  $\text{hom}^1(M, M') \neq 0$  it follows that  $\phi(M) \leq \phi(M') + 1 < \phi(a^{j+1})$ . Since  $\phi(a^m) < \phi(M)$ , it follows from Remark 3.8 (a) that  $m \leq j$ .

If  $m = j$ , then  $\phi(M) < \phi(a^{m+1})$  and we obtain the second system of inequalities.

If  $m < j$ , then we show that the first system of inequalities in (109) holds. Now  $\phi(M) < \phi(a^{j+1})$  and  $\text{hom}^1(a^{j+1}, a^m) \neq 0$ , hence  $\phi(a^{m+1}) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(a^m) + 1$  and  $\phi(M) < \phi(a^{j+1}) \leq \phi(a^m) + 1$ . We have also  $M' \in \sigma^{ss}$  and by  $\text{hom}^1(M, M') \neq 0$  it follows that  $\phi(M) \leq \phi(M') + 1$ . From  $\text{hom}(M', a^m) \neq 0$  it follows  $\phi(M') \leq a^m$ . These arguments together with (111) imply:

$$(112) \quad \begin{aligned} & \phi(a^m) - 1 < \phi(a^{m+1}[-1]) < \phi(a^m); \\ & \phi(a^m) - 1 < \phi(M[-1]) \leq \phi(M') \leq \phi(a^m); \quad \phi(M[-1]) < \phi(a^m). \end{aligned}$$

In (83) we have  $Z(a^m) - Z(a^{m+1}) = Z(\delta)$ , therefore it remains to show that:

$$(113) \quad \arg_{(\phi(a^m)-1, \phi(a^m))}(Z(\delta)) > \phi(M) - 1.$$

From the second row of (112) and (86) we see that  $Z(\delta)$  and  $Z(M[-1])$  both lie in the half-plane<sup>9</sup>  $Z(a^m)_-^c$ . In (54) we have also  $Z(M') = Z(\delta) + Z(M[-1])$ , therefore the vector  $Z(M')$  is in  $Z(a^m)_-^c$  as well, hence by  $Z(M') = Z(\delta) + Z(M[-1])$  it follows that the inequality (113) is equivalent to  $\phi(M') > \phi(M[-1])$ . Therefore it remains to show that  $\phi(M') \neq \phi(M[-1])$ . Indeed, on one hand  $\phi(M[-1]) = \phi(M')$  implies  $\arg_{(\phi(a^m)-1, \phi(a^m))}(Z(\delta)) = \phi(M')$ . On the other hand,  $\sigma \in (M', a^j, a^{j+1})$ ,  $m < j$  and (111) imply  $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(a^m) + 1 < \phi(M) + 1 \leq \phi(M') + 2$ . Thus, we see that  $\phi(M[-1]) = \phi(M')$  implies  $Z(a^{j+1}) \in Z(\delta)_-^c$ . However, from the first inequality in (112) and Corollary 3.18 (d) it follows that  $Z(a^{j+1}) \in Z(\delta)_+^c$ , which is a contradiction, and (113) follows.

So far we showed that  $\sigma \in (a^m, a^{m+1}, M) \cap Z$  implies (109). We show now converse inclusion.

We assume first that the second system of inequalities in (109) holds. In particular  $\sigma \in (a^m, a^{m+1}, M)$ . By the inequality  $\phi(M) < \phi(a^{m+1})$  we can apply Proposition 2.9 (b), hence the triangle (82) implies that  $b^{m+1} \in \sigma^{ss}$ ,  $\phi(M) \leq \phi(b^{m+1}) \leq \phi(a^{m+1})$ , and  $Z(M) + Z(a^{m+1}) = Z(b^{m+1})$ . We have in (109) also  $\phi(a^{m+1}) - 1 < \phi(M) < \phi(a^{m+1})$  and it follows that  $\phi(M) < \phi(b^{m+1}) < \phi(a^{m+1})$ . If the inequality  $\phi(a^{m+1}) > \phi(a^m) + 1$  holds, then due to  $\phi(M) < \phi(b^{m+1}) < \phi(a^{m+1})$  and  $\phi(a^m) < \phi(M)$  hold we obtain that  $\sigma \in (a^m, b^{m+1}, a^{m+1}) \subset Z$  (see table (99)).

Thus, we can assume that  $\phi(a^{m+1}[-1]) \leq \phi(a^m)$  and combining with the inequalities  $\phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(M)$ ,  $\phi(a^m) < \phi(M)$  (given in (109)) we get  $\phi(M[-1]) < \phi(a^{m+1}[-1]) \leq \phi(a^m) < \phi(M)$ . Now it is easy to show with the help of Corollaries 3.7 and 3.11 that  $(a^m, a^{m+1}[-1], M[-1])$  is a  $\sigma$ -triple (see Definition 2.8). Combining the triangles (81) and (82) we get the following sequence:

$$(114) \quad \begin{array}{ccccccc} 0 & \xrightarrow{\quad} & M[-1] & \xrightarrow{\quad} & b^{m+1}[-1] & \xrightarrow{\quad} & M' \\ & \searrow \text{dashed} & \nearrow & \searrow \text{dashed} & \nearrow & \searrow \text{dashed} & \nearrow \\ & & M[-1] & & a^{m+1}[-1] & & a^m \end{array}$$

The conditions of Lemma 2.11 (b) are satisfied with the triple  $(a^m, a^{m+1}[-1], M[-1])$  and the diagram above. Therefore  $M' \in \sigma^{ss}$  and  $\phi(M') < \phi(a^m)$ .

If  $\phi(a^{m+1}[-1]) = \phi(a^m)$ , then it follows that  $\phi(M') + 1 < \phi(a^{m+1})$ , and recalling that we have also  $\phi(a^m) < \phi(a^{m+1})$  we see that  $\sigma \in (M', a^m, a^{m+1}) \subset Z$  (see table (100)).

Therefore we can assume that  $\phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(a^m) < \phi(M)$ . We will show in this case that  $\sigma \in (M', a^j, a^{j+1})$  for some big enough  $j$ . From Proposition 2.2 and Remark 3.9 it follows that  $\{a^{j+1}\}_{j \in \mathbb{Z}} \subset \sigma^{ss}$ . From  $\phi(a^m) < \phi(a^{m+1}) < \phi(a^m) + 1$  and Corollary 3.18 (b) and (d) we see that  $\phi(a^j) < \phi(a^{j+1})$  and  $Z(a^j) \in Z(\delta)_+^c$  for each  $j$  (recall also Remark 3.8 (a)). We will show that for big enough  $j$  we have  $\phi(M') + 1 < \phi(a^j)$  and then from table (100) we obtain  $\sigma \in (M', a^j, a^{j+1}) \subset Z$ .

<sup>9</sup>The notation  $Z(a^m)_-^c$  is explained in (2).

Now we have  $\phi(a^m[-1]) < \phi(M[-1]) < \phi(a^{m+1}[-1]) < \phi(a^m)$ . Recalling that  $Z(\delta) = Z(a^m) + Z(a^{m+1}[-1])$ , we see that we can choose  $t \in \mathbb{R}$  so that  $t < \phi(a^m) < \phi(M) < \phi(a^{m+1}) < t + 1$  and  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$ . Since  $\text{hom}^1(a^j, a^m) \neq 0$ ,  $\text{hom}(a^m, a^j) \neq 0$  for  $j > m + 1$  and by Corollary 3.18 (b), we have  $\phi(a^m) < \phi(a^j) < \phi(a^m) + 1$  for  $j > m + 1$ . These inequalities together with the incidence  $Z(a^j) \in Z(\delta)_+^c$  imply that  $\arg_{(t, t+1)}(Z(a^j)) = \phi(a^j)$  for  $j > m + 1$  (see Remark 2.1 (c)). Now the formula (88) in Corollary 3.19 gives us the following equality:

$$(115) \quad \lim_{j \rightarrow \infty} \phi(a^j) = t + 1.$$

We showed that  $\phi(M') < \phi(a^m)$  (see below (114)) and we have also  $\phi(a^m) < \phi(M)$ . Using  $\text{hom}^1(M, M') \neq 0$  we see that  $\phi(a^m[-1]) < \phi(M') < \phi(a^m)$ . We showed also that  $t < \phi(a^m) < \phi(M) < \phi(a^{m+1}) < t + 1$ . Since  $Z(M) + Z(M') = Z(\delta) = |Z(\delta)| \exp(i\pi t)$ , it follows that  $Z(M') \in Z(\delta)_-^c$  and  $\phi(M') < t$ . By (115) we get the desired  $\phi(a^j) > \phi(M') + 1$  for big  $j$ .

So, we showed that the second system of inequalities in (109) implies that  $\sigma \in (a^m, a^{m+1}, M) \cap Z$ . We show now that the first system in (109) implies  $\sigma \in (a^m, a^{m+1}, M) \cap Z$  as well. Assume that  $a^m, a^{m+1}, M \in \sigma^{ss}$  and that these inequalities hold. They contain the inequalities defining  $(a^m, a^{m+1}, M)$  (see table (100)), therefore we obtain  $\sigma \in (a^m, a^{m+1}, M)$  immediately. Furthermore the first two inequalities show that  $(a^m, a^{m+1}[-1], M[-1])$  is a  $\sigma$ -triple. The conditions of Lemma 2.11 (a) are satisfied with the triple  $(a^m, a^{m+1}[-1], M[-1])$  and the diagram (114). Therefore  $M' \in \sigma^{ss}$  and  $\phi(M') < \phi(a^m)$ . By  $\text{hom}^1(M, M') \neq 0$  we can write  $\phi(a^m) - 1 < \phi(M[-1]) \leq \phi(M') < \phi(a^m)$ , hence by (86) we see that  $Z(\delta), Z(M[-1]), Z(M') \in Z(a^m)_-^c$ . Let us denote  $t = \arg_{(\phi(a^m)-1, \phi(a^m))}(Z(\delta))$ . The third inequality in (109) is the same as  $t > \phi(M) - 1$ . Combining these arguments with the equality  $Z(M') = Z(\delta) + Z(M[-1])$  we write:

$$(116) \quad \phi(a^m[-1]) < \phi(M[-1]) < \phi(M') < t < \phi(a^m).$$

We will show that  $\sigma \in (M', a^j, a^{j+1})$  for some big enough  $j$ . We have  $\phi(a^m) < \phi(a^{m+1}) < \phi(a^m) + 1$  (the first inequality in (109)), which by Proposition 2.2 and Remark 3.9 implies that  $\{a^{j+1}\}_{j \in \mathbb{Z}} \subset \sigma^{ss}$ , and by Corollary 3.18 (b), (d) implies that  $\phi(a^j) < \phi(a^{j+1})$  and  $Z(a^j) \in Z(\delta)_+^c$  for each  $j$ . Since  $\text{hom}^1(a^j, a^m) \neq 0$ ,  $\text{hom}(a^m, a^j) \neq 0$  for  $j > m + 1$ , we have  $\phi(a^m) < \phi(a^j) < \phi(a^m) + 1$  for  $j > m + 1$ . These inequalities together with the incidences  $Z(a^j) \in Z(\delta)_+^c$ ,  $\phi(a^m) \in (t, t + 1)$  imply that  $\arg_{(t, t+1)}(Z(a^j)) = \phi(a^j)$  for  $j > m + 1$  (see Remark 2.1 (c)). Now the formula (88) in Corollary 3.19 leads to (115) again. Therefore by (116) we see that  $\phi(a^j) > \phi(M') + 1$  for big enough  $j$ . It follows that  $\sigma \in (M', a^j, a^{j+1}) \subset Z$  (see table (100)).

The first part of the lemma is shown. It is easy now to show that the intersection is contractible. The intersection in question is the same as  $(a^m, a^{m+1}[-1], M[-1]) \cap Z$ . Let us denote  $\mathcal{E} = (a^m, a^{m+1}[-1], M[-1])$ . We have a homeomorphism  $f_{\mathcal{E}|\Theta_{\mathcal{E}}} : \Theta_{\mathcal{E}} \rightarrow f_{\mathcal{E}}(\Theta_{\mathcal{E}})$  (see (10), (8)). The proved description of  $Z \cap \Theta_{\mathcal{E}}$  by the inequalities (109) shows that  $f_{\mathcal{E}}(Z \cap \Theta_{\mathcal{E}})$  is union of two sets. The first set after permutation of the coordinates in  $\mathbb{R}^6$  is the same as the set considered in Corollary A.3, hence it is also contractible. The second is obviously contractible. Furthermore, one easily shows that the intersection of these two sets is  $\mathbb{R}_{>0}^3 \times \{\phi_0 - 1 < \phi_2 < \phi_1 < \phi_0\}$ , which is contractible as well. Now by Remark A.6 it follows that  $f_{\mathcal{E}}(Z \cap \Theta_{\mathcal{E}})$  is contractible, therefore  $Z \cap \Theta_{\mathcal{E}}$  is contractible as well. Recalling that  $Z$  and  $(a^m, a^{m+1}, M)$  are contractible and applying again Remark A.6 we deduce that  $(a^m, a^{m+1}, M) \cup Z$  is contractible. The lemma is proved.  $\square$

**Corollary 6.4.** *The set  $\mathfrak{T}_a^{st}$  is contractible.*

*Proof.* Recall that  $\mathfrak{T}_a^{st} = Z \cup \bigcup_{j \in \mathbb{N}} (a^j, a^{j+1}, M)$  (see (108)). We will show that  $Z \cup \bigcup_{j=0}^n (a^{m-j+1}, a^{m-j}, M)$  is contractible for each  $m \in \mathbb{Z}$  and each  $n \in \mathbb{N}$ . Then the corollary follows from Remark A.6.

Assume that for some  $n \in \mathbb{N}$  the set  $Z \cup \bigcup_{j=0}^n (a^{m-j+1}, a^{m-j}, M)$  is contractible for each  $m \in \mathbb{Z}$ . We have shown this statement for  $n = 0$  in Lemma 6.3, and now we make induction assumption. Take any  $m \in \mathbb{N}$  and consider  $Z \cup \bigcup_{j=0}^{n+1} (a^{m-j+1}, a^{m-j}, M) = \left( Z \cup \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M) \right) \cup (a^m, a^{m+1}, M)$ . By the induction assumption  $Z \cup \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M)$  and  $(a^m, a^{m+1}, M)$  are contractible. We will show now that the intersection of these sets is contractible as well and then by Remark A.6 we obtain that the union  $Z \cup \bigcup_{j=0}^{n+1} (a^{m-j+1}, a^{m-j}, M)$  is contractible. Indeed, we have

$$(117) \quad \left( Z \cup \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M) \right) \cap (a^m, a^{m+1}, M) = \\ ((a^m, a^{m+1}, M) \cap Z) \cup \left( (a^m, a^{m+1}, M) \cap \bigcup_{j=1}^{n+1} (a^{m-j+1}, a^{m-j}, M) \right).$$

Using Lemmas 6.3 and 5.4 we deduce that the considered intersection consists of the stability conditions for which  $a^m, a^{m+1}, M$  are semi-stable and some of the two systems of inequalities in (109) or the system

$$(109) \quad \begin{array}{l} \phi(a^m) < \phi(a^{m+1}) < \phi(a^m) + 1 \\ \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array} \quad \text{holds. Since the first system in (109) implies}$$

the last system we deduce that the intersection (117) is described by the inequalities:

$$(118) \quad \begin{array}{l} \phi(a^m) < \phi(a^{m+1}) < \phi(a^m) + 1 \\ \phi(a^m) < \phi(M) \\ \phi(a^{m+1}) < \phi(M) + 1 \end{array} \quad \text{or} \quad \begin{array}{l} \phi(a^m) < \phi(a^{m+1}) \\ \phi(a^m) < \phi(M) \\ \phi(M) < \phi(a^{m+1}) < \phi(M) + 1 \end{array}.$$

Now analogous arguments as in the last paragraph of the proof of Lemma 6.3 show that the intersection (117) is contractible. The corollary follows.  $\square$

We pass to the proof that  $\mathfrak{T}_b^{st}$  is contractible. Let us denote

$$(119) \quad W = (M, \_, \_) \cup \bigcup_{p \in \mathbb{Z}} (b^p, a^p, a^{p+1}).$$

Corollary 5.7 and Lemmas 6.1, 6.2 imply (recall Remark A.6) that  $W$  is contractible. From (96) and (98) we see that:

$$(120) \quad \mathfrak{T}_b^{st} = W \cup (\_, \_, M') = W \cup \bigcup_{m \in \mathbb{Z}} (b^m, b^{m+1}, M').$$

The proof of the next Lemma 6.5 is analogous to the proof of Lemma 6.3):

**Lemma 6.5.** *The set  $(b^m, b^{m+1}, M') \cap W$  consists of the stability conditions  $\sigma$  for which  $b^m, b^{m+1}, M'$  are semistable and*

$$(121) \quad \begin{array}{l} \phi(b^m) - 1 < \phi(b^{m+1}[-1]) < \phi(b^m) \\ \phi(b^m) - 1 < \phi(M'[-1]) < \phi(b^m) \\ \arg_{(\phi(b^m)-1, \phi(a^m))}(Z(b^m) - Z(b^{m+1})) > \phi(M') - 1 \end{array} \quad \text{or} \quad \begin{array}{l} \phi(M') < \phi(b^{m+1}) \\ \phi(b^m) < \phi(b^{m+1}) \\ \phi(b^m) < \phi(M') \\ \phi(b^{m+1}) < \phi(M') + 1 \end{array}.$$

It follows that  $(b^m, b^{m+1}, M') \cap W$  and  $(b^m, b^{m+1}, M') \cup W$  are contractible.

*Proof.* In (106) we have that  $(b^m, b^{m+1}, M') \cap (b^j, a^j, b^{j+1}) = \emptyset$  for  $j \neq m$ . Therefore (recall (119))

$$(122) \quad (b^m, b^{m+1}, M') \cap W = (b^m, b^{m+1}, M') \cap ((b^m, a^m, b^{m+1}) \cup (M, \_, \_)).$$

We show first the inclusion  $\subset$ . Assume that  $\sigma \in (b^m, b^{m+1}, M') \cap W$ . Then  $b^m, b^{m+1}, M'$  are semistable and from table (100) we see that

$$(123) \quad \begin{array}{l} \phi(b^m) < \phi(b^{m+1}) \\ \phi(b^m) < \phi(M') \\ \phi(b^{m+1}) < \phi(M') + 1 \end{array}.$$

Taking into account (122) we consider two cases.

If  $\sigma \in (b^m, a^m, b^{m+1})$ , then  $a^m \in \sigma^{ss}$  and  $\phi(a^m) < \phi(b^{m+1})$  (see table (99)). From  $\text{hom}(M', a^m) \neq 0$  (see (71)) it follows  $\phi(M') \leq \phi(a^m) < \phi(b^{m+1})$  and we get the second system in (121).

If  $\sigma \in (M, b^j, b^{j+1})$ , then  $M, b^j, b^{j+1} \in \sigma^{ss}$  and in table (100) we see that  $\phi(M) + 1 < \phi(b^{j+1})$  and  $\phi(b^j) < \phi(b^{j+1})$ . From  $\text{hom}^1(M', M) \neq 0$  it follows that  $\phi(M') \leq \phi(M) + 1 < \phi(b^{j+1})$ . Since we have  $\phi(b^m) < \phi(M')$ , Remark 3.8 (a) implies that  $m \leq j$ .

If  $m = j$ , then  $\phi(M') < \phi(b^{m+1})$  and we obtain the second system of inequalities.

If  $m < j$ , then we will show that the first system of inequalities in (121) holds. Now  $\phi(M') < \phi(b^{j+1})$  and  $\text{hom}^1(b^{j+1}, b^m) \neq 0$ , hence  $\phi(b^{m+1}) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(b^m) + 1$ ,  $\phi(M') < \phi(b^{j+1}) \leq \phi(b^m) + 1$ . We have also  $M \in \sigma^{ss}$  and by  $\text{hom}^1(M', M) \neq 0$  and  $\text{hom}(M, b^m) \neq 0$  it follows that  $\phi(M') \leq \phi(M) + 1$  and  $\phi(M) \leq b^m$ . These arguments together with (123) imply

$$(124) \quad \begin{array}{l} \phi(b^m) - 1 < \phi(b^{m+1}[-1]) < \phi(b^m) \\ \phi(b^m) - 1 < \phi(M'[-1]) \leq \phi(M) \leq \phi(b^m); \quad \phi(M'[-1]) < \phi(b^m) \end{array}.$$

Due to (83), to show the first system in (121) it remains to derive the following inequality:

$$(125) \quad \arg_{(\phi(b^m)-1, \phi(b^m))}(Z(\delta)) > \phi(M') - 1.$$

From (124) we see that  $Z(\delta)$  and  $Z(M'[-1])$  both lie in the half-plane<sup>10</sup>  $Z(b^m)_-^c$ . In (54) we have also  $Z(M) = Z(\delta) + Z(M'[-1])$ , therefore the vector  $Z(M)$  is in  $Z(b^m)_-^c$  as well. Now the equality  $Z(M) = Z(\delta) + Z(M'[-1])$  implies that (125) is equivalent to  $\phi(M) > \phi(M'[-1])$ . Hence we have to show that  $\phi(M) \neq \phi(M'[-1])$ . Indeed, on one hand  $\phi(M'[-1]) = \phi(M)$  implies  $\arg_{(\phi(b^m)-1, \phi(b^m))}(Z(\delta)) = \phi(M)$ . On the other hand,  $\sigma \in (M, b^j, b^{j+1})$ ,  $m < j$  and (123) imply  $\phi(M) + 1 < \phi(b^{j+1}) \leq \phi(b^m) + 1 < \phi(M') + 1 \leq \phi(M) + 2$ . Thus, we see that  $\phi(M'[-1]) = \phi(M)$  implies  $Z(b^{j+1}) \in Z(\delta)_-^c$ . However, from the first inequality in (124) and Corollary 3.18 (d) it follows that  $Z(b^{j+1}) \in Z(\delta)_+^c$ , which is a contradiction, and (125) follows.

So far we showed the inclusion  $\subset$ . We show now the inverse inclusion  $\supset$ .

<sup>10</sup>The notation  $Z(b^m)_-^c$  is explained in (2).



We assume first that the second system of inequalities in (121) holds. In particular  $\sigma \in (b^m, b^{m+1}, M')$ . By the inequality  $\phi(M') < \phi(b^{m+1})$  we can apply Proposition 2.9 (b), hence the short exact sequence (81) implies that  $a^m \in \sigma^{ss}$  and  $Z(M') + Z(b^{m+1}) = Z(a^m)$ . We have also  $\phi(b^{m+1}) - 1 < \phi(M') < \phi(b^{m+1})$ , and it follows that  $\phi(M') < \phi(a^m) < \phi(b^{m+1})$ . If the inequality  $\phi(b^{m+1}) > \phi(b^m) + 1$  holds, then recalling that  $\phi(b^m) < \phi(M')$  we obtain that  $\sigma \in (b^m, a^m, b^{m+1}) \subset W$  (see table (99)).

Therefore we reduce to the inequality  $\phi(b^{m+1}[-1]) \leq \phi(b^m)$ . Combining with  $\phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(M')$  and  $\phi(b^m) < \phi(M')$ , we can write  $\phi(M'[-1]) < \phi(b^{m+1}[-1]) \leq \phi(b^m) < \phi(M')$  and then  $(b^m, b^{m+1}[-1], M'[-1])$  is a  $\sigma$ -triple (see Definition 2.8). Combining the triangles (81) and (82) we obtain the following sequence of triangles in  $\mathcal{T}$ :

$$(126) \quad \begin{array}{ccccccc} 0 & \xrightarrow{\quad} & M'[-1] & \xrightarrow{\quad} & a^m[-1] & \xrightarrow{\quad} & M \\ & \swarrow \text{dashed} & \nwarrow \text{dashed} & \swarrow \text{dashed} & \nwarrow \text{dashed} & \swarrow \text{dashed} & \\ & M'[-1] & & b^{m+1}[-1] & & b^m & \end{array}.$$

The conditions of Lemma 2.11 (b) are satisfied with the triple  $(b^m, b^{m+1}[-1], M'[-1])$  and the diagram above. Therefore  $M \in \sigma^{ss}$  and  $\phi(M) < \phi(b^m)$ .

If  $\phi(b^{m+1}[-1]) = \phi(b^m)$ , then we have also  $\phi(M) + 1 < \phi(b^{m+1})$ , and recalling that we have also  $\phi(b^m) < \phi(b^{m+1})$  we see that  $\sigma \in (M, b^m, b^{m+1}) \subset W$  (see table (100)).

Therefore we can assume that  $\phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(b^m) < \phi(M')$ . We will that  $\sigma \in (M, b^j, b^{j+1})$  for some big  $j$  in this case. Proposition 2.2 and Remark 3.9 ensure that  $\{b^{j+1}\}_{j \in \mathbb{Z}} \subset \sigma^{ss}$ . From  $\phi(b^m) < \phi(b^{m+1}) < \phi(b^m) + 1$  and Corollary 3.18 (b) and (d) we see that  $\phi(b^j) < \phi(b^{j+1})$  and  $Z(b^j) \in Z(\delta)_+^c$  for each  $j$ . Now to show that  $\sigma \in (M, b^j, b^{j+1}) \subset W$  it is enough to derive  $\phi(M) + 1 < \phi(b^j)$  for big enough  $j$  (see table (100)).

Since we have  $\phi(b^m[-1]) < \phi(M'[-1]) < \phi(b^{m+1}[-1]) < \phi(b^m)$  and  $Z(\delta) = Z(b^m) + Z(b^{m+1}[-1])$ , we see that we can choose  $t \in \mathbb{R}$  so that  $t < \phi(b^m) < \phi(M') < \phi(b^{m+1}) < t + 1$  and  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$ . Since  $\text{hom}^1(b^j, b^m) \neq 0$ ,  $\text{hom}(b^m, b^j) \neq 0$  for  $j > m + 1$  and by Corollary 3.18 (b), we have  $\phi(b^m) < \phi(b^j) < \phi(b^m) + 1$  for  $j > m + 1$ . These inequalities together with the incidence  $Z(b^j) \in Z(\delta)_+^c$  imply that  $\arg_{(t, t+1)}(Z(b^j)) = \phi(b^j)$  for  $j > m + 1$  (see Remark 2.1 (c)). The formula (88) in Corollary 3.19 gives us the following:

$$(127) \quad \lim_{j \rightarrow \infty} \phi(b^j) = t + 1.$$

We showed that  $\phi(M) < \phi(b^m)$  (see below (126)) and we have also  $\phi(b^m) < \phi(M')$ . From  $\text{hom}^1(M', M) \neq 0$  we derive  $\phi(b^m[-1]) < \phi(M) < \phi(b^m)$ . We showed also that  $t < \phi(b^m) < \phi(M') < \phi(b^{m+1}) < t + 1$ . Since  $Z(M) + Z(M') = Z(\delta) = |Z(\delta)| \exp(i\pi t)$ , it follows that  $Z(M) \in Z(\delta)_-^c$  and  $\phi(M) < t$ . Now (127) ensures that  $\phi(b^j) > \phi(M) + 1$  for big enough  $j$ . So far we showed that the second system of inequalities in (121) implies  $\sigma \in (b^m, b^{m+1}, M') \cap W$ .

We pass to the first system of inequalities in (121). So assume that  $b^m, b^{m+1}, M' \in \sigma^{ss}$  and that these inequalities hold. They contain the inequalities defining  $(b^m, b^{m+1}, M')$  (see table (100)), hence  $\sigma \in (b^m, b^{m+1}, M')$ . Furthermore, the first two inequalities show that  $(b^m, b^{m+1}[-1], M'[-1])$  is a  $\sigma$ -triple and that the conditions of Lemma 2.11 (a) are satisfied with this triple and the diagram (126). Therefore  $M \in \sigma^{ss}$  and  $\phi(M) < \phi(b^m)$ . By  $\text{hom}^1(M', M) \neq 0$  we can write  $\phi(b^m) - 1 < \phi(M'[-1]) \leq \phi(M) < \phi(b^m)$  (we use also (121)), hence  $Z(\delta), Z(M'[-1]), Z(M) \in Z(b^m)_-^c$ . Let us denote  $t = \arg_{(\phi(b^m)-1, \phi(b^m))}(Z(\delta))$ . The third inequality in (121) is the same as  $t > \phi(M') - 1$ . Combining these arguments with the equality  $Z(M) = Z(\delta) + Z(M'[-1])$  we deduce that:

$$(128) \quad \phi(b^m[-1]) < \phi(M'[-1]) < \phi(M) < t < \phi(b^m).$$

We will show that  $\sigma \in (M, b^j, b^{j+1})$  for some big enough  $j$ . We have  $\phi(b^m) < \phi(b^{m+1}) < \phi(b^m) + 1$  (the first inequality in (121)), which by Proposition 2.2 and Remark 3.9 implies that  $\{b^{j+1}\}_{j \in \mathbb{Z}} \subset \sigma^{ss}$ , and by Corollary 3.18 (b), (d) implies that  $\phi(b^j) < \phi(b^{j+1})$  and  $Z(b^j) \in Z(\delta)_+^c$  for each  $j$ . Using Remark 3.8 one easily shows that  $\phi(b^m) < \phi(b^j) < \phi(b^m) + 1$  for  $j > m + 1$ . These inequalities together with the incidences  $Z(b^j) \in Z(\delta)_+^c$ ,  $\phi(b^m) \in (t, t + 1)$  imply that  $\arg_{(t, t+1)}(Z(b^j)) = \phi(b^j)$  for  $j > m + 1$  (see Remark 2.1 (c)). The formula (88) in Corollary 3.19 leads to (127) again. Now (128) implies that  $\phi(b^j) > \phi(M) + 1$  for big  $j$ , hence  $\sigma \in (M, b^j, b^{j+1}) \subset W$  (see table (100)).

The arguments showing that  $(b^m, b^{m+1}, M') \cap W$  and  $(b^m, b^{m+1}, M') \cup W$  are contractible are as in the last paragraph of the proof of Lemma 6.3. The lemma is proved.  $\square$

**Corollary 6.6.** *The set  $\mathfrak{T}_b^{st}$  is contractible.*

*Proof.* Recall that  $\mathfrak{T}_b^{st} = W \cup \bigcup_{j \in \mathbb{N}} (b^j, b^{j+1}, M')$  (see (120)). Using Lemmas 6.5 and 5.4 one shows by induction that  $W \cup \bigcup_{j=0}^n (b^{m-j+1}, b^{m-j}, M')$  is contractible for each  $m \in \mathbb{Z}$  and each  $n \in \mathbb{N}$  (see the proof of Corollary 6.4 for details). Then the corollary follows from Remark A.6.  $\square$

## 7. CONNECTING $\mathfrak{T}_a^{st}$ AND $\mathfrak{T}_b^{st}$ BY $(\_, M, \_)$ AND $(\_, M', \_)$

Due to the union (94), to prove Theorem 1.2 it remains to connect the contractible non-intersecting pieces  $\mathfrak{T}_a^{st}$ ,  $\mathfrak{T}_b^{st}$  by  $(\_, M, \_)$  and  $(\_, M', \_)$ , and to show that in this procedure the contractibility is preserved. We describe first the building blocks of  $(\_, M, \_)$  and  $(\_, M', \_)$  by Proposition 2.7:

From the list of triples  $\mathfrak{T}$  given in Corollary 3.12 we see that (see also (92)):

$$(129) \quad (\_, M \_) = \bigcup_{q \in \mathbb{Z}} (a^q, M, b^{q+1}) \quad (\_, M' \_) = \bigcup_{q \in \mathbb{Z}} (b^q, M', a^q).$$

We apply Proposition 2.7 to the triples  $(a^p, M, b^{p+1})$  and  $(b^q, M', a^q)$ . Using Corollaries 3.7 and 3.11 one shows that the coefficients  $\alpha, \beta, \gamma$  defined in (29) are  $\alpha = 0$ ,  $\beta = \gamma = -1$  in both the cases. Thus we obtain the formulas in the first and the second column of table (130) for the contractible subsets  $(a^p, M, b^{p+1}) \subset \text{Stab}(D^b(\mathcal{T}))$  and  $(b^q, M', a^q) \subset \text{Stab}(D^b(\mathcal{T}))$ , respectively:

$$(130) \quad \begin{array}{|c|c|} \hline (a^p, M, b^{p+1}) & (b^q, M', a^q) \\ \hline \left\{ \begin{array}{l} \phi(a^p) < \phi(M) + 1 \\ a^p, M, b^{p+1} \in \sigma^{ss} : \begin{array}{l} \phi(a^p) < \phi(b^{p+1}) \\ \phi(M) < \phi(b^{p+1}) \end{array} \end{array} \right\} & \left\{ \begin{array}{l} \phi(b^q) < \phi(M') + 1 \\ b^q, M', a^q \in \sigma^{ss} : \begin{array}{l} \phi(b^q) < \phi(a^q) \\ \phi(M') < \phi(a^q) \end{array} \end{array} \right\} \\ \hline \end{array}$$

**Remark 7.1.**  $(a^p, M, b^{p+1}[-1])$ ,  $(b^q, M', a^q[-1])$  are Ext-exceptional triples (satisfy (a) in Def. 2.8).

In some steps of this section, when we need to show that certain exceptional objects are semi-stable, the tools in Section 2 are not efficient enough. For these cases we prove Lemmas 7.2 and 7.3 below. The relation  $R \dashrightarrow (S, E)$  between a  $\sigma$ -regular object  $R$  and an exceptional pair generated by it (introduced in [9]) is utilized in the proof of these lemmas.

**Lemma 7.2.** *Let  $a^m \notin \sigma^{ss}$  and  $t = \phi_-(a^m)$ , then one of the following holds:*

- (a)  $a^j \in \sigma^{ss}$  for some  $j < m - 1$  and  $t = \phi(a^j) + 1$ ; (b)  $a^j \in \sigma^{ss}$  for some  $m < j$  and  $t = \phi(a^j)$ ;
- (c)  $b^j \in \sigma^{ss}$  for some  $j < m$  and  $t = \phi(b^j) + 1$ ; (d)  $b^j \in \sigma^{ss}$  for some  $m < j$  and  $t = \phi(b^j)$ ;
- (e)  $M \in \sigma^{ss}$  and  $t = \phi(M) + 1$ .

*Proof.* Recall that any  $X \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\}$  is a trivially coupling object (see [9, after Lemma 10.28]). Since  $a^m[k] \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\}$ , where  $k \in \{0, -1\}$ , from  $a^m \notin \sigma^{ss}$  and [9, Lemma 6.3] it follows that  $a^m[k]$  is a  $\sigma$ -regular object, hence  $a^m$  is a  $\sigma$ -regular object. Therefore we have  $R \dashrightarrow (S, E)$  for some exceptional pair  $(S, E)$  (see [9, Section 5]). We will need the following two properties of the exceptional object  $S$ . The first is  $S \in \sigma^{ss}$ ,  $\phi(S) = \phi_-(a^m)$  (see [9, formula (42) after Definition 5.2]). The second property is  $\text{hom}(a^m, S) \neq 0$ , which follows from [9, (c) after formula (19)] and the way  $S$  was chosen (see [9, Definition 5.2]). Recall that there exists at most one nonzero element in the family  $\{\text{hom}^k(a^m, X)\}_{k \in \mathbb{Z}}$  for any  $X \in \mathcal{T}_{exc}$  (Corollary 3.11). By Remark 3.6 we have  $S \in \{a^j[k], b^j[k] : j \in \mathbb{Z}, k \in \mathbb{Z}\} \cup \{M[k], M'[k]; k \in \mathbb{Z}\}$ . Obviously  $S \neq a^m[k]$  (since  $a^m \notin \sigma^{ss}$  and  $S \in \sigma^{ss}$ ).

Now we will use the property  $\text{hom}(a^m, S) \neq 0$  and Corollary 3.7 to prove the lemma. By  $\text{hom}^*(a^m, M') = 0$  (see (71)) we exclude also the case  $S = M'[k]$ . It remains to consider the following cases (one of them must appear):

If  $S = a^j[k]$  for some  $j \neq m$  and  $k \in \mathbb{Z}$ , then by (75) we see that either  $j < m - 1$  and  $k = 1$ , or  $m < j$  and  $k = 0$ .

If  $S = b^j[k]$  for some  $j \in \mathbb{Z}$  and  $k \in \mathbb{Z}$ , then by (74) we see that either  $j < m$  and  $k = 1$ , or  $m < j$  and  $k = 0$ .

If  $S = M[k]$  for some  $k \in \mathbb{Z}$ , then by (72) we get  $k = 1$ . The lemma follows.  $\square$

**Lemma 7.3.** *Let  $b^m \notin \sigma^{ss}$  and  $t = \phi_-(b^m)$ , then one of the following holds:*

- (a)  $a^j \in \sigma^{ss}$  for some  $j < m - 1$  and  $t = \phi(a^j) + 1$ ; (b)  $a^j \in \sigma^{ss}$  for some  $m \leq j$  and  $t = \phi(a^j)$ ;
- (c)  $b^j \in \sigma^{ss}$  for some  $j < m - 1$  and  $t = \phi(b^j) + 1$ ; (d)  $b^j \in \sigma^{ss}$  for some  $m < j$  and  $t = \phi(b^j)$ ;
- (e)  $M' \in \sigma^{ss}$  and  $t = \phi(M') + 1$ .

*Proof.* By the same arguments as in the proof of Lemma 7.2 one shows that  $\text{hom}(b^m, S) \neq 0$  and  $\phi(S) = t$  for some  $S \in \sigma^{ss} \cap (\{a^j[k], M, M' : j \in \mathbb{Z}, k \in \mathbb{Z}\} \cup \{b^j[k]; k \in \mathbb{Z}, j \in \mathbb{Z}, j \neq m\})$ . Now we will use Corollaries 3.7 and 3.11. By  $\text{hom}^*(b^m, M) = 0$  (see (72)) we exclude the case  $S = M[k]$ . It remains to consider the following cases (one of them must appear):

If  $S = a^j[k]$  for some  $j \in \mathbb{Z}$  and  $k \in \mathbb{Z}$ , then by (73) we see that either  $j < m - 1$  and  $k = 1$ , or  $m \leq j$  and  $k = 0$ .

If  $S = b^j[k]$  for some  $j \neq m$  and  $k \in \mathbb{Z}$ , then by (76) we see that either  $j < m - 1$  and  $k = 1$ , or  $m < j$  and  $k = 0$ .

If  $S = M'[k]$  for some  $k \in \mathbb{Z}$ , then by (72) we get  $k = 1$ . The lemma follows.  $\square$

Lemmas 7.4 and 7.5 put together the arguments which ensure semi-stability, necessary later in the analysis of the intersections  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_{a/b}^{st}$  and  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$ .

**Lemma 7.4.** *Let  $\sigma \in (a^p, M, b^{p+1})$  and let the following inequality hold:*

$$(131) \quad \phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1}) .$$

*Then we have the following:*

- (a)  $a^{p+1} \in \sigma^{ss}$  and  $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$ .
- (b) If in addition to (131) we have  $\phi(a^p) < \phi(M)$ , then  $\sigma \in (a^p, a^{p+1}, M)$ .
- (c) If in addition to (131) we have

$$(132) \quad \phi(b^{p+1}) - 1 < \phi(a^p) < \phi(b^{p+1}) ,$$

*then  $M' \in \sigma^{ss}$  and  $\phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p)$ .*

(d) If (131), (132) hold and  $\phi(M') < \phi(M)$ , then  $\sigma \in (a^j, a^{j+1}, M)$  for some  $j \in \mathbb{Z}$ .

*Proof.* (a) We apply Proposition 2.9 (b) to the triple  $(a^p, M, b^{p+1})$  and since  $a^{p+1}[-1]$  is in the extension closure of  $M, b^{p+1}[-1]$  (by (82)) it follows that  $a^{p+1} \in \sigma^{ss}$ . The inequality  $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$  follows from the given inequality (131) and  $Z(a^{p+1}[-1]) = Z(M) + Z(b^{p+1}[-1])$ .

(b) From the given inequalities it follows that  $\phi(a^p) < \phi(b^{p+1})$ . We have also  $\phi(b^{p+1}) < \phi(a^{p+1}) < \phi(M) + 1$  from (a). Therefore we obtain the inequalities  $\phi(a^p) < \phi(a^{p+1})$ ,  $\phi(a^p) < \phi(M)$ ,  $\phi(a^{p+1}) < \phi(M) + 1$ , which means that  $\sigma \in (a^p, a^{p+1}, M)$  (see table (100)).

(c) Follows from Lemma 2.12 applied to the Ext-triple  $(a^p, M, b^{p+1}[-1])$  and the triangle (81).

(d) Now by the given inequalities and (c) we have  $\phi(b^{p+1}) - 1 < \phi(M') < \phi(M) < \phi(b^{p+1})$ . Recalling that  $Z(\delta) = Z(M') + Z(M)$ , we see that we can choose  $t \in \mathbb{R}$  with  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$  and  $\phi(M') < t < \phi(M) < \phi(b^{p+1}) < t + 1$ . If  $\phi(a^p) < \phi(M)$ , then (d) follows from (b).

So let  $\phi(M) \leq \phi(a^p)$ . Since we have also  $\phi(a^p) < \phi(b^{p+1})$ , we obtain  $t < \phi(M) \leq \phi(a^p) < \phi(b^{p+1}) < t + 1$ . Now Corollary 3.19 shows that  $\{Z(a^j), Z(b^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$  and that (88), (87) hold for both the sequences  $\{Z(a^j)\}_{j \in \mathbb{Z}}$  and  $\{Z(b^j)\}_{j \in \mathbb{Z}}$ . From (a) we see that  $\phi(b^{p+1}) < \phi(a^{p+1}) < \phi(M) + 1$ , hence  $t < \phi(a^{p+1}) < t + 2$ , which combined with  $Z(a^{p+1}) \in Z(\delta)_+^c$  implies that  $\phi(a^{p+1}) < t + 1$ . Thus we obtain the inequalities  $t < \phi(M) \leq \phi(a^p) < \phi(b^{p+1}) < \phi(a^{p+1}) < t + 1$ .

From (88) we see that there exists  $N \in \mathbb{Z}$ ,  $N < p$  such that  $t < \arg_{(t,t+1)}(Z(a^j)) < \phi(M)$  for  $j < N$ . We will show below that  $a^j \in \sigma^{ss}$  for  $j < N$ . Then (d) follows. Indeed, assume that  $a^j \in \sigma^{ss}$  for each  $j < N$ . Then by (75) and Corollary 3.18 (a) it follows that  $\phi(a^{p+1}) - 1 < \phi(a^j) < \phi(a^{p+1})$  for  $j < N$ , therefore  $t - 1 < \phi(a^j) < t + 1$ , which combined with  $Z(a^j) \in Z(\delta)_+^c$  implies that  $\arg_{(t,t+1)}(Z(a^j)) = \phi(a^j)$ . Putting the last equality in (87) and in  $\arg_{(t,t+1)}(Z(a^j)) < \phi(M)$  we get  $\phi(a^{j-1}) < \phi(a^j) < \phi(M)$  which by table (100) implies that  $\sigma \in (a^{j-1}, a^j, M)$ .

Suppose  $a^j \notin \sigma^{ss}$  for some  $j < N$ . From Remark 3.9 we know that  $a^j$  is in the extension closure of  $a^p, a^{p+1}[-1]$ . It follows that  $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)]$  and then  $\phi(a^{p+1}) - 1 \leq \phi_-(a^j)$  (recall the paragraph after (5)). We will use Lemma 7.2 and show that each of the five cases given there leads to a contradiction. We first derive (133). The inequalities  $\phi(a^p) - 1 < \phi(a^{p+1}) - 1 < \phi(M) \leq \phi(a^p)$  can be used due to the previous steps. Therefore we have  $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)] \subset \mathcal{P}(\phi(a^p) - 1, \phi(a^p))$ . Using  $\phi(a^p) \in (t, t + 1)$ ,  $Z(a^j) \in Z(\delta)_+^c$ , and Remark 2.1 (c) we get:  $\arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^j)) = \arg_{(t,t+1)}(Z(a^j))$ . Now by Remark 2.1 (a) we get  $\phi_-(a^j) < \arg_{(t,t+1)}(Z(a^j))$  and by our choice of  $N$  we have  $\arg_{(t,t+1)}(Z(a^j)) < \phi(M)$ . We combine these facts in the following inequalities:

$$(133) \quad \phi(a^{p+1}) - 1 \leq \phi_-(a^j) < \arg_{(t,t+1)}(Z(a^j)) < \phi(M) \leq \phi(a^p) < \phi(a^{p+1}).$$

One of the cases in Lemma 7.2 must appear. In case (a) we have  $\phi_-(a^j) = \phi(a^k) + 1$  for some  $k < j - 1$ , hence by (133) it follows  $\text{hom}^1(a^p, a^k) = 0$ , which contradicts (75) and  $j < N < p$ .

In case (b):  $\phi_-(a^j) = \phi(a^k)$  for some  $k > j$ . It follows that  $\phi(a^k) = \arg_{(t,t+1)}(Z(a^k))$  (see Remark 2.1 (c)), hence by (133) and (86) we get  $\arg_{(t,t+1)}(Z(a^k)) < \arg_{(t,t+1)}(Z(a^j))$ , which contradicts (87).

In cases (c) and (d) we have  $\phi_-(a^j) = \phi(b^k)$  or  $\phi(b^k) + 1$  for some  $k \in \mathbb{Z}$ , and then (133) implies  $\text{hom}(M, b^k) = 0$ , which contradicts (71).

Case (e) in Lemma 7.2 and (133) imply that  $\phi(M) + 1 < \phi(M)$  and we proved the lemma.  $\square$

**Lemma 7.5.** Let  $\sigma \in (a^p, M, b^{p+1})$  and let the following inequality hold:

$$(134) \quad \phi(a^p) - 1 < \phi(M) < \phi(a^p) .$$

Then we have the following:

- (a)  $b^p \in \sigma^{ss}$  and  $\phi(M) < \phi(b^p) < \phi(a^p)$ .
- (b) If in addition to (134) we have  $\phi(M) + 1 < \phi(b^{p+1})$ , then  $\sigma \in (M, b^p, b^{p+1})$ .
- (c) If in addition to (134) we have

$$(135) \quad \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p),$$

then  $M' \in \sigma^{ss}$  and  $\phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p)$ .

- (d) If (134), (135) hold and  $\phi(M) < \phi(M')$ , then  $\sigma \in (b^j, b^{j+1}, M')$  for some  $j \in \mathbb{Z}$  or  $\sigma \in (M, b^p, b^{p+1})$ .
- (e) If (134), (135) hold and  $\phi(M) = \phi(M')$ , then  $\sigma \in (a^j, M, b^{j+1})$  for each  $j < p$ .

*Proof.* (a) We apply Proposition 2.9 (a) to the triple  $(a^p, M, b^{p+1})$  and since  $b^p$  is in the extension closure of  $M, a^p$  (by (82)) it follows that  $b^p \in \sigma^{ss}$  and  $\phi(M) \leq \phi(b^p) \leq \phi(a^p)$ . The inequality  $\phi(M) < \phi(b^p) < \phi(a^p)$  follows from the given inequality (134) and  $Z(b^p) = Z(M) + Z(a^p)$ .

(b) From the given inequalities we have  $\phi(a^p) < \phi(M) + 1 < \phi(b^{p+1})$ . In (a) we showed that  $\phi(M) < \phi(b^p) < \phi(a^p)$ . Therefore we obtain the inequalities  $\phi(M) < \phi(b^p)$ ,  $\phi(M) + 1 < \phi(b^{p+1})$ ,  $\phi(b^p) < \phi(b^{p+1})$ , which means that  $\sigma \in (M, b^p, b^{p+1})$  (see table (100)).

(c) Follows from Lemma 2.12 applied to the Ext-triple  $(a^p, M, b^{p+1}[-1])$  and the triangle (81).

(d) Now by the given inequalities and (c) we have  $\phi(a^p) - 1 < \phi(M) < \phi(M') < \phi(a^p)$ . Recalling that  $Z(\delta) = Z(M') + Z(M)$ , we see that we can choose  $t \in \mathbb{R}$  with  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$  and  $\phi(M) < t < \phi(M') < \phi(a^p) < \phi(M) + 1$ . If  $\phi(M) + 1 < \phi(b^{p+1})$ , then we apply (b).

So, let  $\phi(b^{p+1}) \leq \phi(M) + 1$ . Since we have also  $\phi(a^p) < \phi(b^{p+1})$ , we obtain  $t < \phi(M') < \phi(a^p) < \phi(b^{p+1}) \leq \phi(M) + 1 < t + 1$ . Now Corollary 3.19 ensures that  $\{Z(a^j), Z(b^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$  and that (88), (87) hold for both the sequences  $\{Z(a^j)\}_{j \in \mathbb{Z}}$  and  $\{Z(b^j)\}_{j \in \mathbb{Z}}$ . From (a) we see that  $\phi(M) < \phi(b^p) < \phi(a^p)$ , hence  $t - 1 < \phi(b^p) < t + 1$ , which combined with  $Z(b^p) \in Z(\delta)_+^c$  implies that  $t < \phi(b^p)$ . Hence we obtain the inequalities

$$(136) \quad t < \phi(b^p) < \phi(a^p) < \phi(b^{p+1}) < t + 1; \quad t < \phi(M') < \phi(a^p) < \phi(b^{p+1}) < t + 1.$$

From (88) and  $t < \phi(M')$  it follows that there exists  $N \in \mathbb{Z}$ ,  $N < p$  such that  $t < \arg_{(t, t+1)}(Z(b^j)) < \phi(M')$  for  $j < N$ . We will show below that  $b^j \in \sigma^{ss}$  for  $j < N$ . Then (d) follows. Indeed, assume that  $b^j \in \sigma^{ss}$  for each  $j < N$ . Then by (76) and Corollary 3.18 (a) it follows that  $\phi(b^{p+1}) - 1 < \phi(b^j) < \phi(b^{p+1})$  for  $j < N$ , and by (136) we get  $t - 1 < \phi(b^j) < t + 1$ , which combined with  $Z(b^j) \in Z(\delta)_+^c$  implies that  $\arg_{(t, t+1)}(Z(b^j)) = \phi(b^j)$ . Putting the last equality in (87) and in  $\arg_{(t, t+1)}(Z(b^j)) < \phi(M')$  we obtain  $\phi(b^{j-1}) < \phi(b^j) < \phi(M')$ , which implies  $\sigma \in (b^{j-1}, b^j, M')$ .

Suppose  $b^j \notin \sigma^{ss}$  for some  $j < N$ . We apply Lemma 7.3 and show that each of the five cases given there leads to a contradiction. We show first (137). From Remark 3.9 we know that  $b^j$  is in the extension closure of  $b^p, b^{p+1}[-1]$  (recall that  $N < p$ ) and we have  $\phi(b^p) - 1 < \phi(b^{p+1}) - 1 < t < \phi(b^p)$  in (136). It follows that  $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)] \subset \mathcal{P}(\phi(b^p) - 1, \phi(b^p))$ . Using  $\phi(b^p) \in (t, t + 1)$ ,  $Z(b^j) \in Z(\delta)_+^c$ , Remark 2.1 (c) and (a), we deduce that  $\arg_{(\phi(b^p)-1, \phi(b^p))}(Z(b^j)) = \arg_{(t, t+1)}(Z(b^j)) > \phi_-(b^j)$ . The incidence  $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)]$  implies  $\phi(b^{p+1}) - 1 \leq \phi_-(b^j)$ , and we get:

$$(137) \quad \phi(b^{p+1}) - 1 \leq \phi_-(b^j) < \arg_{(t, t+1)}(Z(b^j)) < \phi(M') < \phi(b^{p+1}).$$

One of the cases in Lemma 7.3 must appear. In cases (a) and (b) we have  $\phi_-(b^j) = \phi(a^k)$  of  $\phi(a^k) + 1$  for some  $k \in \mathbb{Z}$ , and then (137) implies  $\text{hom}(M', a^k) = 0$ , which contradicts (71).

In case (c) we have  $\phi_-(b^j) = \phi(b^k) + 1$  for some  $k < j - 1$ , and (137) implies that  $\text{hom}^1(b^{p+1}, b^k) = 0$ , which contradicts (76) and  $k < j - 1 < p - 1$ .

In case (d) we have  $\phi_-(b^j) = \phi(b^k)$  for some  $k > j$ . From  $Z(b^k) \in Z(\delta)_+^c$ , (137), and  $\phi(b^{p+1}) \in (t, t+1)$  it follows that  $\phi(b^k) = \arg_{(t,t+1)}(Z(b^k))$ . Hence (137) and (86) imply  $\arg_{(t,t+1)}(Z(b^k)) < \arg_{(t,t+1)}(Z(b^j))$ , which contradicts  $k > j$  and (87).

Case (e) in Lemma 7.3 and (137) imply that  $\phi(M') + 1 < \phi(M')$ . We proved completely part (d) of the lemma.

(e) Now by the given inequalities we have  $\phi(a^p) - 1 < \phi(M) = \phi(M') < \phi(a^p)$ . Recalling that  $Z(\delta) = Z(M') + Z(M)$ , we see that  $t = \phi(M) = \phi(M')$  satisfies  $Z(\delta) = |Z(\delta)| \exp(i\pi t)$  and  $t < \phi(a^p) < t+1$ . From (a) we get  $t < \phi(b^p) < \phi(a^p) < t+1$ . Now we can apply Corollary 3.20, which besides  $\{Z(a^j), Z(b^j)\}_{j \in \mathbb{Z}} \subset Z(\delta)_+^c$  and formulas (87), (88) gives us the inequalities (89).

We extend the inequality  $t < \phi(b^p) < \phi(a^p) < t+1$  to (138) as follows. We already have that  $a^p, b^p, b^{p+1} \in \sigma^{ss}$ . In (135) is given that  $\phi(a^p) < \phi(b^{p+1})$ . From  $\text{hom}^1(b^{p+1}, M')$  (see (72)) it follows  $\phi(b^{p+1}) \leq t+1$  and from  $Z(b^{p+1}) \in Z(\delta)_+^c$  we see that  $\phi(b^{p+1}) < t+1 = \phi(M) + 1$ . We have also  $\phi(M) < \phi(b^{p+1})$  (due to  $\sigma \in (a^p, M, b^{p+1})$ ). Therefore  $\phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1})$  and from Lemma 7.4 (a) we get  $a^{p+1} \in \sigma^{ss}$  and  $\phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < \phi(M)$ . Thus, we derive:

$$(138) \quad \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^{p+1}) - 1 < t < \phi(b^p) < \phi(a^p) < \phi(b^{p+1}) < \phi(a^{p+1}) < t+1.$$

We will show below that  $a^j$  and  $b^j$  are semi-stable for each  $j < p$ . We claim that this implies  $\sigma \in (a^j, M, b^{j+1})$  for  $j < p$ . Indeed, assume that  $a^j, b^j \in \sigma^{ss}$  for each  $j < p$ . Then by (75), (76) we get  $\phi(a^{p+1}) - 1 \leq \phi(a^j) \leq \phi(a^{p+1})$  and  $\phi(b^{p+1}) - 1 \leq \phi(b^j) \leq \phi(b^{p+1})$ , which combined with  $t < \phi(b^{p+1}) < \phi(a^{p+1}) < t+1$  and  $Z(a^j), Z(b^j) \in Z(\delta)_+^c$  implies that  $\phi(a^j), \phi(b^j) \in (t, t+1)$ , in particular  $\phi(a^j) = \arg_{(t,t+1)}(Z(a^j))$  and  $\phi(b^j) = \arg_{(t,t+1)}(Z(b^j))$  for each  $j < p$ . The last two equalities hold also for  $j = p$  by (138). Putting these equalities in (89) we get that  $\phi(a^j) < \phi(b^{j+1})$  for each  $j < p$ . Thus, we obtain  $\phi(M) < \phi(a^j) < \phi(b^{j+1}) < \phi(M) + 1$  for each  $j < p$ , which by table (130) gives  $\sigma \in (a^j, M, b^{j+1})$ .

Suppose that  $b^j \notin \sigma^{ss}$  for some  $j < p$ . Remark 3.9 asserts that  $b^j$  is in the extension closure of  $b^p, b^{p+1}[-1]$ , therefore  $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)]$ , and hence  $\phi(b^{p+1}) - 1 \leq \phi_-(b^j)$ ,  $\phi_+(b^j) \leq \phi(b^p)$ . Due to (138) we can write  $b^j \in \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^p)] \subset \mathcal{P}[\phi(a^p) - 1, \phi(a^p)]$ . Using  $\phi(a^p) \in (t, t+1)$ ,  $Z(b^j) \in Z(\delta)_+^c$  and Remark 2.1 (c) we conclude that  $\arg_{(\phi(a^p)-1, \phi(a^p))}(Z(b^j)) = \arg_{(t,t+1)}(Z(b^j))$ . Now using Remark 2.1 (a), we obtain:

$$(139) \quad \phi(b^{p+1}) - 1 \leq \phi_-(b^j) < \arg_{(t,t+1)}(Z(b^j)) < \phi_+(b^j) \leq \phi(b^p) < \phi(b^{p+1}).$$

We use Lemma 7.3 and show that each of the five cases given there leads to a contradiction.

Case (a) ensures  $\phi_-(b^j) = \phi(a^k) + 1$  for some  $k < j-1$  and (139) implies that  $\text{hom}^1(b^{p+1}, a^k) = 0$ , which contradicts (73) (now  $k < p$ ).

Case (b) ensures  $\phi_-(b^j) = \phi(a^k)$  for some  $k \geq j$ , and then (139) and  $Z(a^k) \in Z(\delta)_+^c$  imply  $\arg_{(t,t+1)}(Z(a^k)) = \phi(a^k)$ , hence by (139) and (86) we get  $\arg_{(t,t+1)}(Z(a^k)) < \arg_{(t,t+1)}(Z(b^j))$ , which contradicts (89) and  $k \geq j$ .

Case (c) ensures  $\phi_-(b^j) = \phi(b^k) + 1$  for some  $k < j-1 < p-1$ , and (139) implies that  $\text{hom}^1(b^{p+1}, b^k) = 0$ , which contradicts (76).

In case (d) we have  $\phi_-(b^j) = \phi(b^k)$  for some  $k > j$ . It follows by  $Z(b^k) \in Z(\delta)_+^c$  and (139) that  $\phi(b^k) = \arg_{(t,t+1)}(Z(b^k))$ , and then (139) gives  $\arg_{(t,t+1)}(Z(b^k)) < \arg_{(t,t+1)}(Z(b^j))$ , which contradicts (87).

In case (e) using (139) we obtain  $\phi(M') + 1 < \phi(b^{p+1})$ , which contradicts (72).

Suppose that  $a^j \notin \sigma^{ss}$  for some  $j < p$ . Since  $a^j$  is in the extension closure of  $a^p$ ,  $a^{p+1}[-1]$  (see Remark 3.9), therefore  $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)]$ , and hence  $\phi_{\pm}(a^j) \in [\phi(a^{p+1}) - 1, \phi(a^p)]$ . Due to (138) we have  $a^j \in \mathcal{P}[\phi(a^{p+1}) - 1, \phi(a^p)] \subset \mathcal{P}[\phi(b^{p+1}) - 1, \phi(b^{p+1})]$  and Remark 2.1 (c) shows that that  $\arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^j)) = \arg_{(t, t+1)}(Z(a^j))$ . Now Remark 2.1 (a) completes the following:

$$(140) \quad \phi(a^{p+1}) - 1 \leq \phi_{-}(a^j) < \arg_{(t, t+1)}(Z(a^j)) < \phi_{+}(a^j) \leq \phi(a^p) < \phi(a^{p+1}).$$

We use Lemma 7.2 to get a contradiction. One of the five cases given there must appear.

In case (a) of Lemma 7.2 we have  $\phi_{-}(a^j) = \phi(a^k) + 1$  for some  $k < j - 1 < p - 1$ , and (140) implies  $\text{hom}^1(a^{p+1}, a^k) = 0$ , which contradicts (75).

Case (b) ensures  $\phi_{-}(a^j) = \phi(a^k)$  for some  $k > j$ . It follows that  $\phi(a^k) = \arg_{(t, t+1)}(Z(a^k))$  (see Remark 2.1 (c)), hence by (140) we get  $\arg_{(t, t+1)}(Z(a^k)) < \arg_{(t, t+1)}(Z(a^j))$ , which contradicts (87).

In case (c) we have  $\phi_{-}(a^j) = \phi(b^k) + 1$  for some  $k < j$  and (140) implies that  $\text{hom}^1(a^{p+1}, b^k) = 0$ , which contradicts (74) (now  $k < p$ ).

Case (d) ensures  $\phi_{-}(a^j) = \phi(b^k)$  for some  $j < k$ , and then  $\arg_{(t, t+1)}(Z(b^k)) = \phi(b^k)$  (see Remark 2.1 (c)), hence by (140) we get  $\arg_{(t, t+1)}(Z(b^k)) < \arg_{(t, t+1)}(Z(a^j))$ , which contradicts (89).

In case (e) we have  $\phi_{-}(a^j) = \phi(M) + 1$ , and (140) implies  $\text{hom}^1(a^{p+1}, M) = 0$ , which contradicts (72). The lemma is proved.  $\square$

Next we glue  $(a^p, M, b^{p+1})$  and  $\mathfrak{T}_a^{st}$ .

**Lemma 7.6.** *For any  $p \in \mathbb{Z}$  the set  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$  consists of the stability conditions  $\sigma$  for which  $a^p, M, b^{p+1}$  are semistable and:*

$$(141) \quad \begin{array}{l} \phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1}) \\ \phi(b^{p+1}) - 1 < \phi(a^p) < \phi(b^{p+1}) \\ \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M) \end{array} \quad \text{or} \quad \begin{array}{l} \phi(a^p) < \phi(M) \\ \phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1}) \end{array}.$$

It follows that  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$  and  $(a^p, M, b^{p+1}) \cup \mathfrak{T}_a^{st}$  are contractible.

*Proof.* We start with the inclusion  $\subset$ . Assume that  $\sigma \in (a^p, M, b^{p+1})$ . Then  $a^p, M, b^{p+1}$  are semi-stable and by table (130) we get

$$(142) \quad \begin{array}{l} \phi(a^p) < \phi(M) + 1 \\ \phi(a^p) < \phi(b^{p+1}) \\ \phi(M) < \phi(b^{p+1}) \end{array}$$

Recalling (95), we see that we have to consider three cases.

If  $\sigma \in (M', a^j, a^{j+1})$ , then  $M', a^j, a^{j+1}$  are semi-stable and from table (100) we see that  $\phi(M') + 1 < \phi(a^{j+1})$ . Since we have also  $\text{hom}^1(b^{p+1}, M')$ ,  $\text{hom}^1(a^{j+1}, M) \neq 0$  (see Corollary 3.7), we obtain  $\phi(b^{p+1}) \leq \phi(M') + 1 < \phi(a^{j+1}) \leq \phi(M) + 1$ , which combined with (142) implies

$$(143) \quad \phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1}) \quad \phi(M') < \phi(M).$$

These non-vanishings and inequalities give also  $\phi(a^p) < \phi(b^{p+1}) \leq \phi(M') + 1 < \phi(a^{j+1})$ . Using Remark 3.8 (a) we deduce that  $p \leq j$ .

We show now that  $\phi(b^{p+1}) < \phi(a^p) + 1$ . If  $j = p$ , then we immediately obtain this by  $\text{hom}^1(b^{p+1}, M') \neq 0$  and  $\phi(M') < \phi(a^p)$  (see table (100)). If  $j > p$ , then  $\text{hom}(b^{p+1}, a^j) \neq 0$  and  $\text{hom}^1(a^{j+1}, a^p) \neq 0$  (see Corollary 3.7) and we can write  $\phi(b^{p+1}) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(a^p) + 1$ .

To obtain the first system of inequalities in (141) it remains to show the third inequality. From the triangle (81) it follows that  $\phi(b^{p+1}) - 1 \leq \phi(M') \leq \phi(a^p)$  and  $Z(M') = Z(a^p) - Z(b^{p+1})$ , now  $\phi(M') = \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M)$  follows from the already proved  $\phi(b^{p+1}) - 1 < \phi(a^p) < \phi(b^{p+1})$  and (143).

If  $\sigma \in (a^m, a^{m+1}, M)$ , then  $a^m, a^{m+1}$  are semistable as well and in table (100) we see that  $\phi(a^m) < \phi(M)$ , which together with the third inequality in (142) imply that  $\phi(a^m) < \phi(b^{p+1})$  and hence  $\text{hom}(b^{p+1}, a^m) = 0$ . By (73) we deduce that  $p \geq m$ .

If  $p = m$ , then we get immediately  $\phi(a^p) < \phi(M)$ . In table (100) we have  $\phi(a^{p+1}) < \phi(M) + 1$  and in Corollary 3.7 we have  $\text{hom}(b^{p+1}, a^{p+1}) \neq 0$ , hence  $\phi(b^{p+1}) < \phi(M) + 1$  and we obtain the second system of inequalities in (141).

If  $p > m$ , then  $\text{hom}^1(b^{p+1}, a^m) \neq 0$  and from the inequalities  $\phi(a^m) < \phi(M)$ ,  $\phi(a^m) < \phi(a^{m+1})$  (due to  $\sigma \in (a^m, a^{m+1}, M)$ ) it follows  $\phi(b^{p+1}) < \phi(M) + 1$  and  $\phi(b^{p+1}) \leq \phi(a^m) + 1 < \phi(a^{m+1}) + 1 \leq \phi(a^p) + 1$ . Recalling (142) we see that we obtained the first two equalities in (141). Hence by Lemma 7.4 (c) we get  $M' \in \sigma^{ss}$  and  $\phi(M') = \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1}))$ . From  $\text{hom}(M', a^m) \neq 0$  it follows  $\phi(M') \leq \phi(a^m) < \phi(M)$  and we obtain the complete first system of inequalities in (141).

If  $\sigma \in (a^m, b^{m+1}, a^{m+1})$ , then  $a^m, b^{m+1}, a^{m+1} \in \sigma^{ss}$  and in table (99) we see that  $\phi(a^m) + 1 < \phi(a^{m+1})$ , hence Lemma 3.10 and  $a^p \in \sigma^{ss}$  imply that  $p = m$  or  $p = m + 1$ . If  $p = m + 1$ , then by (142) we obtain  $\phi(a^m) + 1 < \phi(a^{m+1}) < \phi(b^{m+2})$ , and hence  $\text{hom}^1(b^{m+2}, a^m) = 0$ , which contradicts (73). Thus, it remains to consider the case  $m = p$ . Now we have  $\phi(a^p) + 1 < \phi(a^{p+1})$  and  $\phi(b^{p+1}) < \phi(a^{p+1})$  (see table (99)), which together with  $\text{hom}^1(a^{p+1}, M) \neq 0$  imply  $\phi(a^p) < \phi(M)$  and  $\phi(b^{p+1}) < \phi(M) + 1$ , hence we obtain the second system in (141). The inclusion  $\subset$  is shown.

We show now the converse inclusion  $\supset$ . Assume that  $a^p, M, b^{p+1}$  are semi-stable and that one of the two systems of inequalities in (141) holds. In both the cases the given inequalities imply the inequalities (142), therefore  $\sigma \in (a^p, M, b^{p+1})$ . If the second system in (141) holds, then by Lemma 7.4 (b) we get  $\sigma \in (a^p, a^{p+1}, M) \subset \mathfrak{T}_a^{st}$ . If the first system in (141) holds, then by Lemma 7.4 (c) and (d) we get  $\sigma \in (a^j, a^{j+1}, M) \subset \mathfrak{T}_a^{st}$  for some  $j \in \mathbb{Z}$ , and the inclusion  $\supset$  is proved as well.

As in the last paragraph of the proof of Lemma 6.3 one shows that the two systems of inequalities in (141) correspond to two contractible sets (the first is contractible by Corollary A.2), and it is easy to show that their intersection is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$ , which is also contractible. Remark A.6 shows that  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$  is contractible. Since  $(a^p, M, b^{p+1})$  and  $\mathfrak{T}_a^{st}$  are both contractible (Proposition 2.7 and Corollary 6.4), Remark A.6 shows that  $(a^p, M, b^{p+1}) \cup \mathfrak{T}_a^{st}$  is contractible as well.  $\square$

**Lemma 7.7.** *For any  $p \in \mathbb{Z}$  the set  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$  consists of the stability conditions  $\sigma$  for which  $a^p, M, b^{p+1}$  are semistable and:*

$$(144) \quad \begin{array}{l} \phi(a^p) - 1 < \phi(M) < \phi(a^p) \\ \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p) \\ \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1})) > \phi(M) \end{array} \quad \text{or} \quad \begin{array}{l} \phi(a^p) - 1 < \phi(M) < \phi(a^p) \\ \phi(M) + 1 < \phi(b^{p+1}) \end{array}.$$

*It follows that  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$  and  $(a^p, M, b^{p+1}) \cup \mathfrak{T}_b^{st}$  are contractible.*



*Proof.* We start with the inclusion  $\subset$ . Assume that  $\sigma \in (a^p, M, b^{p+1})$ . Then  $a^p, M, b^{p+1}$  are semi-stable and by table (130) we get

$$(145) \quad \begin{aligned} \phi(a^p) &< \phi(M) + 1 \\ \phi(a^p) &< \phi(b^{p+1}) \\ \phi(M) &< \phi(b^{p+1}) \end{aligned}$$

Recalling (96), we see that we have to consider three cases.

If  $\sigma \in (M, b^j, b^{j+1})$ , then  $M, b^j, b^{j+1}$  are semi-stable and from table (100) we see that  $\phi(M) < \phi(b^j)$  and  $\phi(M) + 1 < \phi(b^{j+1})$ , hence  $\phi(a^p) < \phi(b^{j+1})$  and  $\text{hom}(b^{j+1}, a^p) = 0$ . From (73) it follows that  $p \leq j$ . If  $j = p$ , then  $\phi(M) + 1 < \phi(b^{p+1})$  and by  $\text{hom}(b^p, a^p) \neq 0$  (see (73)) we get  $\phi(M) < \phi(a^p)$ , which implies the second system in (144). It remains to consider the case  $p < j$ .

In this case  $\text{hom}^1(b^{j+1}, a^p) \neq 0$  (see (73)) and we obtain  $\phi(M) + 1 < \phi(b^{j+1}) \leq \phi(a^p) + 1$ , which combined with (145) implies  $\phi(a^p) - 1 < \phi(M) < \phi(a^p)$ . On the other hand, we have  $\phi(b^j) < \phi(b^{j+1})$  (see table (100)), and by  $p < j$  we can write  $\phi(b^{p+1}) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(a^p) + 1$ , which combined with (145) implies  $\phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p)$ . Now we can use Lemma 7.5 (c) to deduce that  $M' \in \sigma^{ss}$  and  $\phi(M') = \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1}))$ . From  $\text{hom}^1(b^{j+1}, M') \neq 0$  and  $\phi(M) + 1 < \phi(b^{j+1})$  it follows that  $\phi(M) < \phi(M')$  and the first system in (144) follows.

If  $\sigma \in (b^m, b^{m+1}, M')$ , then  $b^m, b^{m+1}, M'$  are semistable and in table (100) we see that  $\phi(b^m) < \phi(M')$ . By  $\text{hom}(M', a^p) \neq 0$  and  $\text{hom}(M, b^m) \neq 0$  (see (71)) we get:

$$(146) \quad \phi(M) \leq \phi(b^m) < \phi(M') \leq \phi(a^p).$$

Whence  $\phi(M) < \phi(a^p)$  and combining with (145) we derive  $\phi(a^p) - 1 < \phi(M) < \phi(a^p)$ . On the other hand, in (146) we have also  $\phi(b^m) < \phi(a^p)$ , and hence  $\text{hom}(a^p, b^m) = 0$ , therefore by (74) we see that  $p \geq m$ . In (146) we have also  $\phi(M) < \phi(M')$ . Taking into account Lemma 7.5 (c), we see that if we show that  $\phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p)$ , then the first system in (144) follows. Since we have  $\phi(a^p) < \phi(b^{p+1})$  (see (145)), it remains to show that  $\phi(b^{p+1}) < \phi(a^p) + 1$ . If  $p = m$ , then from table (100) we obtain  $\phi(b^{p+1}) < \phi(M') + 1$  and the inequality in question follows from  $\phi(M') \leq \phi(a^p)$ . If  $m < p$ , then  $\text{hom}^1(b^{p+1}, b^m) \neq 0$  and we get  $\phi(b^{p+1}) \leq \phi(b^m) + 1 < \phi(a^p) + 1$  (see (146)).

If  $\sigma \in (b^m, a^m, b^{m+1})$ , then  $b^m, a^m, b^{m+1} \in \sigma^{ss}$  and in table (99) we see that  $\phi(b^m) + 1 < \phi(b^{m+1})$ , hence Lemma 3.10 and  $b^{p+1} \in \sigma^{ss}$  imply that  $p = m$  or  $p = m - 1$ . If  $p = m - 1$ , then by (145) we obtain  $\phi(a^{m-1}) + 1 < \phi(b^m) + 1 < \phi(b^{m+1})$ , and hence  $\text{hom}^1(b^{m+1}, a^{m-1}) = 0$ , which contradicts (73). Therefore we have  $m = p$ . Now we have  $\phi(b^p) + 1 < \phi(b^{p+1})$  and  $\phi(b^p) < \phi(a^p)$  (see table (99)), which together with  $\text{hom}(M, b^p) \neq 0$  imply  $\phi(M) + 1 < \phi(b^{p+1})$  and  $\phi(M) < \phi(a^p)$ , hence the second system in (144) follows. Thus we showed the inclusion  $\subset$ .

We show now the inverse inclusion  $\supset$ . Assume that  $a^p, M, b^{p+1}$  are semi-stable and that one of the two systems of inequalities in (144) holds. In both the cases the given inequalities imply the inequalities (145), therefore  $\sigma \in (a^p, M, b^{p+1})$ . If the second system in (144) holds, then by Lemma (7.5) (b) we get  $\sigma \in (M, b^p, b^{p+1}) \subset \mathfrak{T}_b^{st}$ . If the first system in (144) holds, then the desired  $\sigma \in \mathfrak{T}_b^{st}$  follows from Lemma (7.5) (c) and (d). The inclusion  $\supset$  is proved as well.

In Corollary 6.6 was shown that  $\mathfrak{T}_b^{st}$  is contractible. The proof that  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$  and  $(a^p, M, b^{p+1}) \cup \mathfrak{T}_b^{st}$  are contractible is as in the last paragraph of Lemma 7.6. The two systems in (144) correspond to contractible subsets of  $(a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$  (the first is contractible by Corollary

A.3). The intersection of these subsets is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$ , which is also contractible. Now we apply Remark A.6 twice and the lemma follows.  $\square$

**Corollary 7.8.** *For any  $p \in \mathbb{Z}$  the set  $\mathfrak{T}_a^{st} \cup (a^p, M, b^{p+1}) \cup \mathfrak{T}_b^{st}$  is contractible.*

*Proof.* In Lemma 7.6 we showed that  $\mathfrak{T}_a^{st} \cup (a^p, M, b^{p+1})$  is contractible. Since  $\mathfrak{T}_a^{st} \cap \mathfrak{T}_b^{st} = \emptyset$  (see Subsection 5.1), it follows that  $(\mathfrak{T}_a^{st} \cup (a^p, M, b^{p+1})) \cap \mathfrak{T}_b^{st} = (a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$ , which is contractible by Lemma 7.7. Now we apply Remark A.6.  $\square$

**Lemma 7.9.** *For any  $q < p$  the set  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  consists of the stability conditions  $\sigma$  for which  $a^p, M, b^{p+1}$  are semistable and:*

$$(147) \quad \begin{aligned} & \phi(a^p) - 1 < \phi(M) < \phi(a^p) \quad \text{or} \quad \phi(a^p) - 1 < \phi(M) < \phi(a^p) \\ & \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p) \quad \text{or} \quad \phi(M) + 1 < \phi(b^{p+1}) \\ & \text{or} \quad \phi(a^p) < \phi(M) \quad \text{or} \quad \begin{aligned} & \phi(b^{p+1}) - 1 < \phi(M) < \phi(b^{p+1}) \\ & \phi(b^{p+1}) - 1 < \phi(a^p) < \phi(b^{p+1}) \end{aligned} \\ & \quad \arg_{(\phi(b^{p+1})-1, \phi(b^{p+1}))}(Z(a^p) - Z(b^{p+1})) < \phi(M) \end{aligned} .$$

*It follows that  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  and  $(a^p, M, b^{p+1}) \cup (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  are contractible.*

*Proof.* We start with the inclusion  $\subset$ . Assume that  $\sigma \in (a^p, M, b^{p+1})$ . Then  $a^p, M, b^{p+1}$  are semi-stable and by table (130) we get

$$(148) \quad \begin{aligned} & \phi(a^p) < \phi(M) + 1 \\ & \phi(a^p) < \phi(b^{p+1}) \\ & \phi(M) < \phi(b^{p+1}) \end{aligned} .$$

If  $\sigma \in (a^q, M, b^{q+1})$  and  $q < p$ , then  $a^q, b^{q+1} \in \sigma^{ss}$  and  $\phi(M) < \phi(b^{q+1})$ ,  $\phi(a^q) < \phi(b^{q+1})$  as well. By (73) we have  $\text{hom}(b^{q+1}, a^p) \neq 0$  and  $\text{hom}^1(b^{p+1}, a^q) \neq 0$ , therefore  $\phi(M) < \phi(b^{q+1}) \leq \phi(a^p)$  and  $\phi(b^{p+1}) \leq \phi(a^q) + 1 < \phi(b^{q+1}) + 1 \leq \phi(a^p) + 1$ . Combining with (148) we obtain the system in the first row and first column in (147).

If  $\sigma \in \mathfrak{T}_a^{st}$ , then by Lemma 7.6 some of the systems on the second row of (147) follows.

If  $\sigma \in \mathfrak{T}_b^{st}$ , then by Lemma 7.7 some of the systems on the first row of (147) follows ((144) implies (147)). So we showed the inclusion  $\subset$ .

We show now the inclusion  $\supset$ . So let  $a^p, M, b^{p+1}$  be semi-stable. If some of the systems on the second row of (147) holds, then by 7.6 it follows that  $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$ . If the system in the first row and second column of (147) holds, then Lemma 7.7 ensures that  $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$ .

Thus, it remains to consider the first system in (147). We assume till the end of the proof that

$$(149) \quad \begin{aligned} & \phi(a^p) - 1 < \phi(M) < \phi(a^p) \\ & \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(a^p) \end{aligned} .$$

Lemma 7.5 (c) ensures that

$$(150) \quad M' \in \sigma^{ss}; \quad \phi(b^{p+1}) - 1 < \phi(M') = \arg_{(\phi(a^p)-1, \phi(a^p))}(Z(a^p) - Z(b^{p+1})) < \phi(a^p).$$

Now we consider three cases.

If  $\phi(M') > \phi(M)$ , then (149) and (150) yield the first system in (144) is satisfied and then Lemma 7.7 says that  $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_b^{st}$ .

If  $\phi(M') < \phi(M)$ , then by  $\text{hom}^1(b^{p+1}, M') \neq 0$  it follows that  $\phi(b^{p+1}) - 1 < \phi(M)$ . Combining this inequality with (149) one easily shows that:

$$(151) \quad \begin{aligned} \phi(b^{p+1}) - 1 &< \phi(M) < \phi(b^{p+1}) \\ \phi(b^{p+1}) - 1 &< \phi(a^p) < \phi(b^{p+1}) \end{aligned} .$$

Having obtained (151) we can use Lemma 7.4 (c) and due to  $\phi(M') < \phi(M)$  we derive the first system in (141). Thus Lemma 7.6 ensures that  $\sigma \in (a^p, M, b^{p+1}) \cap \mathfrak{T}_a^{st}$ .

Finally, if  $\phi(M) = \phi(M')$ , then due to (149) we can apply Lemma 7.5 (e), which says that  $\sigma \in (a^p, M, b^{p+1}) \cap (a^q, M, b^{q+1})$  (recall that  $q < p$ ). So far we showed the first part of the lemma.

We explain now, using the obtained representation through the systems of inequalities (147), that  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  is contractible. The four systems correspond to four open subsets of  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  (see the last paragraph of the proof of Lemma 6.3). We denote these subsets by  $S_{11}, S_{12}, S_{21}, S_{22}$ , where  $S_{ij}$  corresponds to the system in the  $i$ -th row and  $j$ -th column of (147). The proved part of the lemma is the equality  $(a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st}) = \bigcup_{1 \leq i, j \leq 2} S_{ij}$ . The subset  $S_{22}$  is contractible by Corollary A.2. The subsets  $S_{11}, S_{12}, S_{21}$  are contractible since they are homeomorphic to convex subsets of  $\mathbb{R}^6$ . For example  $S_{11}$  is homeomorphic to

$$\mathbb{R}_{>0}^3 \times \left\{ (\phi_0, \phi_1, \phi_2) \in \mathbb{R}^3 : \begin{array}{l} \phi_0 - 1 < \phi_1 < \phi_0 \\ \phi_0 - 1 < \phi_2 - 1 < \phi_0 \end{array} \right\} .$$

One easily shows that  $S_{11} \cap S_{12}$  is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$ , hence it is contractible, and by Remark A.6 we deduce that  $S_{11} \cup S_{12}$  is contractible. Note that in  $S_{12}$  we have  $\phi(M) + 1 < \phi(b^{p+1})$  and in  $S_{22}$  we have  $\phi(M) + 1 > \phi(b^{p+1})$ , therefore  $S_{12} \cap S_{22} = \emptyset$ . Hence  $S_{22} \cap (S_{11} \cup S_{12}) = S_{22} \cap S_{11}$ . One easily shows that  $S_{22} \cap S_{11}$  is homeomorphic to:

$$(152) \quad \mathbb{R}_{>0}^3 \times \left\{ (\phi_0, \phi_1, \phi_2) \in \mathbb{R}^3 : \begin{array}{l} r_i > 0 \\ \phi_2 - 1 < \phi_1 < \phi_0 < \phi_2 \\ \arg_{(\phi_2-1, \phi_2)}(r_0 \exp(i\pi\phi_0) - r_2 \exp(i\pi\phi_2)) < \phi_1 \end{array} \right\} ,$$

which by Corollary A.5 is contractible as well. Thus, we see that  $S_{22} \cap (S_{11} \cup S_{12})$  is contractible, therefore by Remark A.6 we see that  $S_{22} \cup S_{11} \cup S_{12}$  is contractible. In  $S_{11}$  and  $S_{12}$  we have  $\phi(M) < \phi(a^p)$  and in  $S_{21}$  we have  $\phi(M) > \phi(a^p)$ , therefore  $S_{21} \cap (S_{22} \cup S_{11} \cup S_{12}) = S_{21} \cap S_{22}$ . On the other hand, one easily shows (by drawing a picture) that the intersection  $S_{21} \cap S_{22}$  is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$ , which is contractible as well, and hence  $S_{21} \cap (S_{22} \cup S_{11} \cup S_{12})$  is contractible. Applying Remark A.6 again ensures that  $S_{21} \cup S_{22} \cup S_{11} \cup S_{12} = (a^p, M, b^{p+1}) \cap (\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st})$  is contractible. In Corollary 7.8 is shown that  $\mathfrak{T}_a^{st} \cup (a^q, M, b^{q+1}) \cup \mathfrak{T}_b^{st}$  is contractible and with one more reference to Remark A.6 we prove the lemma.  $\square$

**Corollary 7.10.** *The set  $(\_, M, \_) \cup \mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}$  is contractible.*

*Proof.* Recall that  $(\_, M, \_) = \bigcup_{q \in \mathbb{Z}} (a^q, M, b^{q+1})$  (see (129)). We will prove that for each  $p \in \mathbb{Z}$  and for each  $k \geq 1$  the set (153) below is contractible, and the corollary follows from Remark A.6:

$$(153) \quad \bigcup_{i=0}^k (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}).$$

In the previous lemma was shown that for  $k = 1$  and any  $p \in \mathbb{Z}$  the set (153) is contractible. Assume that for some  $k \geq 1$  this set is contractible for each  $p \in \mathbb{Z}$ . Take now any  $p \in \mathbb{Z}$ . We have

$$(154) \quad \bigcup_{i=0}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) = (a^p, M, b^{p+1}) \cup \left( \bigcup_{i=1}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) \right).$$

Proposition 2.7 and the induction assumption say that the two components on RHS of (154) are contractible. Since the intersection analyzed in Lemma 7.9 does not depend on  $q$ , we can write:

$$(a^p, M, b^{p+1}) \cap \left( \bigcup_{i=1}^{k+1} (a^{p-i}, M, b^{p+1-i}) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}) \right) = (a^p, M, b^{p+1}) \cap ((a^{p-1}, M, b^p) \cup (\mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st})),$$

which by Lemma 7.9 is contractible. Now Remark A.6 ensures that that (154) is contractible.  $\square$

The next step is to glue  $(\_, M, \_) \cup \mathfrak{T}_a^{st} \cup \mathfrak{T}_b^{st}$  and  $(b^p, M', a^p)$ . This is done in several substeps: Lemmas 7.11, 7.12, 7.13, 7.14, which lead to Corollary 7.15. In the next two lemmas we prove inclusions in only one direction not equality of sets.

**Lemma 7.11.** *Let  $p \in \mathbb{Z}$ . If  $\sigma \in (b^p, M', a^p) \cap \mathfrak{T}_b^{st}$ , then  $b^p, M', a^p$  are semistable and:*

$$(155) \quad \begin{array}{l} \phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(a^p) - 1 < \phi(b^p) < \phi(a^p) \end{array} \quad \text{or} \quad \begin{array}{l} \phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ \phi(b^p) < \phi(M') \end{array}.$$

*Proof.* In table (130) we see that  $b^p, M', a^p$  are semi-stable and:

$$(156) \quad \begin{array}{l} \phi(b^p) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) \\ \phi(M') < \phi(a^p) \end{array}$$

Recalling (96), we see that we have to consider three cases.

If  $\sigma \in (M, b^j, b^{j+1})$ , then  $M, b^j, b^{j+1}$  are semi-stable and from table (100) we see that  $\phi(M) < \phi(b^j)$  and  $\phi(M) + 1 < \phi(b^{j+1})$ . By  $\text{hom}^1(a^p, M) \neq 0$  and  $\text{hom}^1(b^{j+1}, M') \neq 0$  (see (72)) we can write  $\phi(a^p) \leq \phi(M) + 1 < \phi(b^{j+1}) \leq \phi(M') + 1$ , therefore (see also (156)) we get

$$(157) \quad \phi(a^p) - 1 < \phi(M') < \phi(a^p).$$

Since  $\phi(b^p) < \phi(a^p) \leq \phi(M) + 1 < \phi(b^{j+1})$ , due to (76) the inequality  $p \leq j$  must hold.

If  $j = p$ , then the inequality  $\phi(M) < \phi(b^p)$  (coming from  $\sigma \in (M, b^j, b^{j+1})$ ) implies  $\phi(a^p) - 1 \leq \phi(M) < \phi(b^p)$  and combining with (156) and (157) we obtain the first system in (155).

If  $p < j$ , then we have  $\text{hom}(a^p, b^j) \neq 0$  (see (74)) and  $\text{hom}^1(b^{j+1}, b^p) \neq 0$ , hence  $\phi(a^p) \leq \phi(b^j) < \phi(b^{j+1}) \leq \phi(b^p) + 1$  and again the first system in (155) follows.

If  $\sigma \in (b^m, b^{m+1}, M')$ , then  $b^m, b^{m+1}, M'$  are semistable and in table (100) we see that  $\phi(b^m) < \phi(M')$ , therefore  $\phi(b^m) < \phi(M') < \phi(a^p)$  and  $\text{hom}(a^p, b^m) = 0$ . From (74) we deduce that  $m \leq p$ .

If  $m = p$ , then the incidence  $\sigma \in (b^m, b^{m+1}, M')$  gives  $\phi(b^p) < \phi(M')$  and  $\phi(b^{p+1}) - 1 < \phi(M')$  (see table (100)), and from  $\text{hom}(a^p, b^{p+1}) \neq 0$  we obtain  $\phi(a^p) - 1 < \phi(M')$ , therefore the second system in (155) holds.

Let  $m < p$ . Then we have  $\phi(b^m) < \phi(b^{m+1})$  and  $\phi(b^m) < \phi(M')$  (see table (100)). Using  $\text{hom}^1(a^p, b^m) \neq 0$  (see (74)) we deduce  $\phi(a^p) \leq \phi(b^m) + 1 < \phi(b^{m+1}) + 1 \leq \phi(b^p) + 1$  and  $\phi(a^p) \leq \phi(b^m) + 1 < \phi(M') + 1$ , which combined with (156) produces the first system in (155).

If  $\sigma \in (b^m, a^m, b^{m+1})$ , then  $b^m, a^m, b^{m+1} \in \sigma^{ss}$  and in table (99) we see that  $\phi(b^m) + 1 < \phi(b^{m+1})$ , hence Lemma 3.10 and  $b^p \in \sigma^{ss}$  imply  $p = m$  or  $p = m + 1$ . If  $p = m + 1$ , then by (156) we obtain  $\phi(b^m) + 1 < \phi(b^{m+1}) < \phi(a^{m+1})$ , and hence  $\text{hom}^1(a^{m+1}, b^m) = 0$ , which contradicts (74). Therefore we have  $m = p$ . In table (99) we see that  $\phi(b^p) + 1 < \phi(b^{p+1})$  and  $\phi(a^p) < \phi(b^{p+1})$ . From  $\text{hom}^1(b^{p+1}, M') \neq 0$  it follows that  $\phi(b^p) + 1 < \phi(b^{p+1}) \leq \phi(M') + 1$  and  $\phi(a^p) < \phi(b^{p+1}) \leq \phi(M') + 1$ . These inequalities together with (156) produce the second system in (155).  $\square$

**Lemma 7.12.** *Let  $p \in \mathbb{Z}$ . If  $\sigma \in (b^p, M', a^p) \cap \mathfrak{T}_a^{st}$ , then  $b^p, M', a^p$  are semistable and:*

$$(158) \quad \begin{array}{l} \phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^p) \end{array} \quad \text{or} \quad \begin{array}{l} \phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ \phi(M') + 1 < \phi(a^p) \end{array}.$$

*Proof.* In table (130) we see that  $b^p, M', a^p$  are semi-stable and:

$$(159) \quad \begin{array}{l} \phi(b^p) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) \\ \phi(M') < \phi(a^p) \end{array}$$

Recalling (95), we see that we have to consider the following three cases.

If  $\sigma \in (M', a^j, a^{j+1})$ , then  $M', a^j, a^{j+1}$  are semi-stable and  $\phi(M') < \phi(a^j)$ ,  $\phi(M') + 1 < \phi(a^{j+1})$ ,  $\phi(a^j) < \phi(a^{j+1})$  (see table (100)). On the other hand  $\phi(b^p) < \phi(M') + 1$ , hence  $\text{hom}(a^{j+1}, b^p) = 0$  and (74) implies that  $p - 1 \leq j$ . If  $p - 1 = j$ , then we have  $\phi(M') + 1 < \phi(a^p)$  and  $\phi(M') < \phi(a^{p-1}) \leq \phi(b^p)$  (see also (74)) and combining with (159) we derive the second system in (158). Let  $p \leq j$ . Then by (74) we have  $\text{hom}^1(a^{j+1}, b^p) \neq 0$  and we can write  $\phi(M') + 1 < \phi(a^{j+1}) \leq \phi(b^p) + 1$  and  $\phi(a^p) \leq \phi(a^j) < \phi(a^{j+1}) \leq \phi(b^p) + 1$ , therefore  $\phi(M') < \phi(b^p)$  and  $\phi(a^p) < \phi(b^p) + 1$ , which combined with (159) amounts to the first system in (158).

If  $\sigma \in (a^m, a^{m+1}, M)$ , then  $a^m, a^{m+1}, M$  are semistable as well and in table (100) we see that  $\phi(a^m) < \phi(M)$ ,  $\phi(a^{m+1}) < \phi(M) + 1$ ,  $\phi(a^m) < \phi(a^{m+1})$ . Since  $\text{hom}(M', a^m) \neq 0$  and  $\text{hom}(M, b^p) \neq 0$ , it follows that  $\phi(M') \leq \phi(a^m) < \phi(M) \leq \phi(b^p)$  and hence (see also (159)):

$$(160) \quad \phi(b^p) - 1 < \phi(M') < \phi(b^p)$$

On the other hand,  $\phi(a^m) < \phi(M)$  and  $\text{hom}(M, b^p) \neq 0$  imply that  $\phi(a^m) < \phi(b^p)$  and  $\text{hom}(b^p, a^m) = 0$ . Now from (73) we deduce that  $m < p$ . If  $m = p - 1$ , then we have  $\phi(a^p) < \phi(M) + 1 \leq \phi(b^p) + 1$ , which together with (160) and (159) amounts to the first system in (158).

If  $m < p - 1$ , then  $\text{hom}^1(a^p, a^m) \neq 0$  and  $\text{hom}(a^{m+1}, b^p) \neq 0$  (see (74)). Therefore we have  $\phi(a^p) \leq \phi(a^m) + 1 < \phi(a^{m+1}) + 1 \leq \phi(b^p) + 1$  and the first system in (158) follows again.

If  $\sigma \in (a^m, b^{m+1}, a^{m+1})$ , then  $a^m, b^{m+1}, a^{m+1} \in \sigma^{ss}$  and in table (99) we see that  $\phi(a^m) + 1 < \phi(a^{m+1})$ , hence Lemma 3.10 and  $a^p \in \sigma^{ss}$  imply  $p = m$  or  $p = m + 1$ . If  $p = m$ , then by (159) we obtain  $\phi(b^m) + 1 < \phi(a^m) + 1 < \phi(a^{m+1})$ , and hence  $\text{hom}^1(a^{m+1}, b^m) = 0$ , which contradicts (74). Thus, it remains to consider the case  $m = p - 1$ . Now we have  $\phi(a^{p-1}) + 1 < \phi(a^p)$  and  $\phi(a^{p-1}) < \phi(b^p)$  (see table (99)), which together with  $\text{hom}(M', a^{p-1}) \neq 0$  imply  $\phi(M') + 1 < \phi(a^p)$  and  $\phi(M') < \phi(b^p)$ , hence we obtain the second system of inequalities in (158).  $\square$

**Lemma 7.13.** *Let  $\sigma \in (b^p, M', a^p)$  and let the following inequality hold:*

$$(161) \quad \phi(b^p) - 1 < \phi(M') < \phi(b^p).$$

*Then we have the following:*

(a)  $a^{p-1} \in \sigma^{ss}$  and  $\phi(M') < \phi(a^{p-1}) < \phi(b^p) < \phi(a^p)$ .

- (b) If in addition to (161) we have  $\phi(M') + 1 < \phi(a^p)$ , then  $\sigma \in (M', a^{p-1}, a^p)$ .  
(c) If in addition to (161) we have  $\phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^p)$ , then  $\sigma \in (a^{p-1}, M, b^p)$ .

*Proof.* (a) We apply Proposition 2.9 (a) to the triple  $(b^p, M', a^p)$  and since  $a^{p-1}$  is in the extension closure of  $M', b^p$  (by (81)) it follows that  $a^{p-1} \in \sigma^{ss}$ ,  $\phi(M') \leq \phi(a^{p-1}) \leq \phi(b^p)$ . The inequality  $\phi(M') < \phi(a^{p-1}) < \phi(b^p)$  follows from the given inequality (161), formula (86) and  $Z(a^{p-1}) = Z(M') + Z(b^p)$ . The inequality  $\phi(b^p) < \phi(a^p)$  follows from  $\sigma \in (b^p, M', a^p)$  (see table (130)).

(b) From the given inequalities and (a) we have  $\phi(M') < \phi(a^{p-1})$ ,  $\phi(M') + 1 < \phi(a^p)$ , and  $\phi(a^{p-1}) < \phi(a^p)$ , then table (100) shows that  $\sigma \in (M', a^{p-1}, a^p)$ .

(c) From Lemma 2.12 applied to the Ext-triple  $(b^p, M', a^p[-1])$  and the triangle (82) we obtain  $M \in \sigma^{ss}$  and  $\phi(a^p) - 1 < \phi(M) < \phi(b^p)$ . In (a) we got  $a^{p-1} \in \sigma^{ss}$  and  $\phi(a^{p-1}) < \phi(a^p)$ , therefore  $\phi(a^{p-1}) < \phi(M) + 1$ . In (a) we have also  $\phi(a^{p-1}) < \phi(b^p)$ . Looking at table (130) we see that  $\sigma \in (a^{p-1}, M, b^p)$ .  $\square$

**Lemma 7.14.** *Let  $\sigma \in (b^p, M', a^p)$  and let the following inequality hold:*

$$(162) \quad \phi(a^p) - 1 < \phi(M') < \phi(a^p).$$

*Then we have the following:*

- (a)  $b^{p+1} \in \sigma^{ss}$  and  $\phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(M')$ .  
(b) If in addition to (162) we have  $\phi(b^p) < \phi(M')$ , then  $\sigma \in (b^p, b^{p+1}, M')$ .  
(c) If in addition to (162) we have  $\phi(a^p) - 1 < \phi(b^p) < \phi(a^p)$ , then  $\sigma \in (a^p, M, b^{p+1})$

*Proof.* (a) We apply Proposition 2.9 (b) to the triple  $(b^p, M', a^p)$  and since  $b^{p+1}[-1]$  is in the extension closure of  $M', a^p[-1]$  (by (81)) it follows that  $b^{p+1} \in \sigma^{ss}$ ,  $\phi(a^p) - 1 \leq \phi(b^{p+1}) - 1 \leq \phi(M')$ . The inequality  $\phi(a^p) - 1 < \phi(b^{p+1}) - 1 < \phi(M')$  follows from the given inequality (162), formula (86), and  $Z(b^{p+1}[-1]) = Z(M') + Z(a^p[-1])$ . The inequality  $\phi(b^p) < \phi(a^p)$  follows from  $\sigma \in (b^p, M', a^p)$ .

(b) From the given inequalities and (a) we have  $\phi(b^p) < \phi(b^{p+1})$ ,  $\phi(b^p) < \phi(M')$  and  $\phi(b^{p+1}) < \phi(M') + 1$ . Now in table (100) we see that  $\sigma \in (b^p, b^{p+1}, M')$ .

(c) From Lemma 2.12 applied to the Ext-triple  $(b^p, M', a^p[-1])$  and the triangle (82) we obtain  $M \in \sigma^{ss}$  and  $\phi(a^p) - 1 < \phi(M) < \phi(b^p)$ . In (a) we showed that  $b^{p+1} \in \sigma^{ss}$  and  $\phi(a^p) < \phi(b^{p+1})$ ,  $\phi(b^p) < \phi(b^{p+1})$ . Now all the conditions determining  $(a^p, M, b^{p+1})$  (given in table (130)) are satisfied, hence  $\sigma \in (a^p, M, b^{p+1})$ .  $\square$

**Corollary 7.15.** *For any  $p \in \mathbb{Z}$  the set  $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  consists of the stability conditions  $\sigma$  for which  $b^p, M', a^p$  are semistable and:*

$$(163) \quad \begin{aligned} & \phi(a^p) - 1 < \phi(M') < \phi(a^p) \quad \text{or} \quad \phi(a^p) - 1 < \phi(M') < \phi(a^p) \\ & \phi(a^p) - 1 < \phi(b^p) < \phi(a^p) \quad \phi(b^p) < \phi(M') \\ & \text{or} \quad \phi(b^p) - 1 < \phi(M') < \phi(b^p) \quad \text{or} \quad \phi(b^p) - 1 < \phi(M') < \phi(b^p) \\ & \phi(b^p) - 1 < \phi(a^p) - 1 < \phi(b^p) \quad \phi(M') + 1 < \phi(a^p) \end{aligned}$$

*It follows that  $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  and  $(b^p, M', a^p) \cup (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  are contractible.*

*Proof.* Due to Lemmas 7.11 and 7.12, to show the inclusion  $\subset$  it remains only to show that the incidence  $\sigma \in (b^p, M', a^p) \cap (\_, M, \_)$  implies some of the systems in (163). Assume that  $\sigma \in (b^p, M', a^p) \cap (a^q, M, b^{q+1})$  for some  $q \in \mathbb{Z}$ . From table (130) we see that  $b^p, M', a^p, a^q, M, b^{q+1}$  are

semi-stable and:

$$(164) \quad \begin{array}{ccc} \phi(b^p) < \phi(M') + 1 & & \phi(a^q) < \phi(M) + 1 \\ \phi(b^p) < \phi(a^p) & \text{and} & \phi(a^q) < \phi(b^{q+1}) \\ \phi(M') < \phi(a^p) & & \phi(M) < \phi(b^{q+1}) \end{array}.$$

If  $p \leq q$ , then the non-vanishings  $\text{hom}(a^p, a^q) \neq 0$ ,  $\text{hom}^1(b^{q+1}, M') \neq 0$ , and  $\text{hom}(M, b^p) \neq 0$  (see Corollary 3.7) together with (164) imply the following inequalities  $\phi(a^p) \leq \phi(a^q) < \phi(M) + 1 \leq \phi(b^p) + 1$  and  $\phi(a^p) \leq \phi(a^q) < \phi(b^{q+1}) \leq \phi(M') + 1$ , which combined with (164) amount to the system in the first row and the first column of (163).

If  $q < p$ , then the non-vanishings  $\text{hom}(M', a^q) \neq 0$ ,  $\text{hom}(b^{q+1}, b^p) \neq 0$ , and  $\text{hom}^1(a^p, M) \neq 0$  together with (164) imply the inequalities  $\phi(M') \leq \phi(a^q) < \phi(b^{q+1}) \leq \phi(b^p)$  and  $\phi(a^p) \leq \phi(M) + 1 < \phi(b^{q+1}) + 1 \leq \phi(b^p) + 1$ . The system in the second row and the first column in (163) follows. So far we showed the inclusion  $\subset$ .

Assume that  $b^p, M', a^p \subset \sigma^{ss}$  and that (163) holds. Each of the systems in (163) contains in it the inequalities of  $(b^p, M', a^p)$  from table (130), hence  $\sigma \in (b^p, M', a^p)$ . Lemmas 7.13 and 7.14 ensure that  $\sigma \in (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  as well and the first part of the corollary follows.

Now the arguments are analogous to those given in the end of the proof of Lemma 7.9.

The four systems in (163) correspond to four open subsets of  $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$ . We denote these subsets by  $S_{11}, S_{12}, S_{21}, S_{22}$ , where  $S_{ij}$  corresponds to the system in the  $i$ -th row and  $j$ -th. The first part of the corollary and Remark A.6 reduce the proof of the last statement to proving that  $\bigcup_{1 \leq i, j \leq 2} S_{ij}$  is contractible.

All of  $S_{11}, S_{12}, S_{21}, S_{22}$  are contractible since they are homeomorphic to convex subsets of  $\mathbb{R}^6$ .

One easily shows that:

- $S_{11} \cap S_{12}$  is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_2 - 1 < \phi_0 < \phi_1 < \phi_2\}$
- $S_{21} \cap (S_{11} \cup S_{12}) = S_{21} \cap S_{11}$  is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_2 - 1 < \phi_1 < \phi_0 < \phi_2\}$
- $S_{22} \cap (S_{11} \cup S_{12} \cup S_{21}) = S_{22} \cap S_{21}$  is homeomorphic to  $\mathbb{R}_{>0}^3 \times \{\phi_0 - 1 < \phi_1 < \phi_2 - 1 < \phi_0\}$ .

Since the obtained subsets of  $\mathbb{R}^6$  are convex, in particular contractible, it follows by Remark A.6 that  $\bigcup_{1 \leq i, j \leq 2} S_{ij}$  is contractible. The corollary follows.  $\square$

We can prove now Theorem 1.2:

**Theorem 7.16.**  $\text{Stab}(D^b(Q))$  is contractible.

*Proof.* Recall that  $\text{Stab}(\mathcal{T}) = \mathfrak{T}_a^{st} \cup (\_, M', \_) \cup (\_, M, \_) \cup \mathfrak{T}_b^{st}$  (see (94)). Recalling (129) we get:

$$(165) \quad \text{Stab}(D^b(Q)) = \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st} \cup \bigcup_{k \in \mathbb{Z}} (b^k, M', a^k).$$

Corollary 7.10 says that  $\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st}$  is contractible and it remains to show that after adding  $\bigcup_{k \in \mathbb{Z}} (b^k, M', a^k)$  the result is still contractible.

We first show that for any two integers  $q > p$  we have:

$$(166) \quad (b^p, M', a^p) \cap (b^q, M', a^q) \subset (b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st}).$$

Assume that  $\sigma \in (b^p, M', a^p) \cap (b^q, M', a^q)$ . Then in table (130) we see that

$$(167) \quad \begin{array}{ccc} \phi(b^p) < \phi(M') + 1 & & \phi(b^q) < \phi(M') + 1 \\ \phi(b^p) < \phi(a^p) & \text{and} & \phi(b^q) < \phi(a^q) \\ \phi(M') < \phi(a^p) & & \phi(M') < \phi(a^q) \end{array}.$$

Since  $p < q$ , we have the non-vanishings  $\text{hom}(a^p, b^q) \neq 0$  and  $\text{hom}^1(a^q, b^p) \neq 0$  (see (74)). We combine with (167) as follows  $\phi(a^p) \leq \phi(b^q) < \phi(a^q) \leq \phi(b^p) + 1$  and  $\phi(a^p) \leq \phi(b^q) < \phi(M') + 1$ , hence  $\phi(a^p) - 1 < \phi(b^p)$  and  $\phi(a^p) - 1 < \phi(M')$ . In (167) we have also  $\phi(b^p) < \phi(a^p)$  and  $\phi(M') < \phi(a^p)$  and the system in the first row and the first column of (163) follows. Therefore by Corollary 7.15 we get  $\sigma \in (b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  and we showed the inclusion (166). This implies that for any  $p \in \mathbb{Z}$  and any  $n \geq 1$  holds the following equality:

$$(b^p, M', a^p) \cap \left( \mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st} \bigcup_{k=1}^n (b^{p+k}, M', a^{p+k}) \right) = (b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st}).$$

In Corollary 7.15 we showed that  $(b^p, M', a^p) \cap (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  and  $(b^p, M', a^p) \cup (\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st})$  are contractible (for any  $p \in \mathbb{Z}$ ). Now using the equality above and Remark A.6 one easily shows by induction that  $\mathfrak{T}_a^{st} \cup (\_, M, \_) \cup \mathfrak{T}_b^{st} \cup \bigcup_{k=0}^n (b^{p+k}, M', a^{p+k})$  is contractible for any  $p \in \mathbb{Z}$  and any  $n \geq 1$ . Applying Remark A.6 again we deduce that the right-hand side of (165) is contractible as well. Therefore  $\text{Stab}(D^b(Q))$  is contractible.  $\square$

#### APPENDIX A. SOME CONTRACTIBLE SUBSETS OF $\mathbb{R}^6$

We prove here that some subsets of  $\mathbb{R}^6$ , which we meet in the proof of Theorem 1.2, are contractible. We start by the following subset

**Lemma A.1.** *The set  $U_>$ , given below, is contractible:*

$$(168) \quad U_> = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{l} r_i > 0 \\ \phi_0 < \phi_1 < \phi_0 + 1 \\ \phi_0 < \phi_2 < \phi_0 + 1 \\ \arg_{(\phi_0, \phi_0+1)}(r_0 \exp(i\pi\phi_0) + r_1 \exp(i\pi\phi_1)) > \phi_2 \end{array} \right\}.$$

The set  $U_<$  defined by the same inequalities, except the last, where we take  $\arg_{(\phi_0, \phi_0+1)}(r_0 \exp(i\pi\phi_0) + r_1 \exp(i\pi\phi_1)) < \phi_2$  is contractible as well.

*Proof.* By drawing a picture one easily shows that:

$$(169) \quad \forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U_> \quad \begin{array}{l} r'_1 \geq r_1 \\ r'_2 > 0 < r'_0 \leq r_0 \\ \phi_0 < \phi_1 \leq \phi'_1 < \phi_0 + 1 \\ \phi_0 < \phi'_2 \leq \phi_2 < \phi_0 + 1 \end{array} \Rightarrow (r'_0, r'_1, r'_2, \phi_0, \phi'_1, \phi'_2) \in U_>.$$

Let  $\gamma : \mathbb{S}^n \rightarrow U$  be a continuous map with  $n \geq 1$ . Denote

$$0 < r_0^{\min} = \min\{r_0(t) : t \in \mathbb{S}^n\}; \quad 0 < r_1^{\max} = \max\{r_1(t) : t \in \mathbb{S}^n\};$$

$$0 < u = \max\{\phi_1(t) - \phi_0(t) : t \in \mathbb{S}^n\} < 1; \quad 0 < v = \min\{\phi_2(t) - \phi_0(t) : t \in \mathbb{S}^n\} < 1;$$

then by (169) for any  $\delta > 0$  and any  $t \in \mathbb{S}^n$ ,  $s \in [0, 1]$  the vector given below lies in  $U_>$ :

$$F(t, s) = \left( \begin{array}{l} r_0(t)(1-s) + sr_0^{\min}, r_1(t)(1-s) + sr_1^{\max}, r_2(t)(1-s) + s\delta, \\ \phi_0(t), \phi_0(t) + (1-s)(\phi_1(t) - \phi_0(t)) + su, \phi_0(t) + (1-s)(\phi_2(t) - \phi_0(t)) + sv \end{array} \right).$$

Hence we obtain a map  $F : \mathbb{S}^n \times [0, 1] \rightarrow U_>$ , whose continuity is obvious. This gives a homotopy from the map  $\gamma$  to the following continuous map:

$$(170) \quad \gamma' : \mathbb{S}^n \rightarrow U_> \quad \gamma'(t) = (r_0^{\min}, r_1^{\max}, \delta, \phi_0(t), \phi_0(t) + u, \phi_0(t) + v)$$



Now we note that:

$$(171) \quad \forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U \quad \forall \delta \in \mathbb{R} \quad (r_0, r_1, r_2, \phi_0 + \delta, \phi_1 + \delta, \phi_2 + \delta) \in U_>$$

Therefore for  $t \in \mathbb{S}^n$ ,  $s \in [0, 1]$  we have

$$G(t, s) = \left( \begin{array}{c} r_0^{\min}, r_1^{\max}, \delta, \phi_0(t) + s(\phi_0(0) - \phi_0(t)), \\ \phi_0(t) + u + s(\phi_0(0) - \phi_0(t)), \phi_0(t) + v + s(\phi_0(0) - \phi_0(t)) \end{array} \right) \in U_{>0}$$

which gives a homotopy from  $\gamma'$  to the constant map from  $\mathbb{S}^n$  to the point  $(r_0^{\min}, r_1^{\max}, \delta, \phi_0(0), \phi_0(0) + u, \phi_0(0) + v) \in U_{>0}$ . Thus, we showed that each continuous map  $\gamma : \mathbb{S}^n \rightarrow U_>$  with  $n \geq 1$  is homotopic to a constant map. If we show that  $U_>$  is connected, then Whitehead theorem ensures that  $U_>$  is contractible. Let  $x = (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U_>$  and  $x' = (r'_0, r'_1, r'_2, \phi'_0, \phi'_1, \phi'_2) \in U_>$ . By (171) we can move continuously  $x'$  in  $U_>$  to  $x'' = (r''_0, r''_1, r''_2, \phi_0, \phi'_1, \phi'_2)$  and now by (169) we can connect  $x, x''$  by a continuous path in  $U_>$ .

The same idea shows that  $U_<$  is contractible, one must permute  $\leq \leftrightarrow \geq$ ,  $\min \leftrightarrow \max$ . The lemma is proved.  $\square$

**Corollary A.2.** *The set  $V$ , given below, is contractible:*

$$(172) \quad V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_2 - 1 < \phi_0 < \phi_2 \\ \phi_2 - 1 < \phi_1 < \phi_2 \\ \arg_{(\phi_2-1, \phi_2)}(r_0 \exp(i\pi\phi_0) - r_2 \exp(i\pi\phi_2)) > \phi_1 \end{array} \right\}.$$

After changing the last inequality to  $\arg_{(\phi_2-1, \phi_2)}(r_0 \exp(i\pi\phi_0) - r_2 \exp(i\pi\phi_2)) < \phi_1$  the set remains contractible.

*Proof.* The assignment  $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_2, a_0, a_1, b_2 - 1, b_0, b_1)$  maps homeomorphically the set  $V$  to the set  $U$  in Lemma A.1.  $\square$

**Corollary A.3.** *The set  $V$ , given below, is contractible:*

$$(173) \quad V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_0 - 1 < \phi_1 < \phi_0 \\ \phi_0 - 1 < \phi_2 < \phi_0 \\ \arg_{(\phi_0-1, \phi_0)}(r_0 \exp(i\pi\phi_0) + r_2 \exp(i\pi\phi_2)) > \phi_1 \end{array} \right\}.$$

*Proof.* The assignment  $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_0, a_2, a_1, -b_0, -b_2, -b_1)$  maps homeomorphically the set  $V$  to the set  $U_<$  in Lemma A.1 (see (4)).  $\square$

**Lemma A.4.** *The set  $U$ , given below, is contractible:*

$$(174) \quad U = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{c} r_i > 0 \\ \phi_2 < \phi_1 < \phi_0 < \phi_2 + 1 \\ \arg_{(\phi_2, \phi_2+1)}(r_0 \exp(i\pi\phi_0) + r_2 \exp(i\pi\phi_2)) < \phi_1 \end{array} \right\}.$$

*Proof.* By drawing a picture one easily checks that:

$$(175) \quad \forall (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in U \quad \begin{array}{c} r'_2 \geq r_2 \\ r'_1 > 0 < r'_0 \leq r_0 \end{array} \Rightarrow (r'_0, r'_1, r'_2, \phi_0, \phi_1, \phi_2) \in U.$$

Let  $\gamma : \mathbb{S}^n \rightarrow U$  be any continuous map with  $n \geq 1$ . Denote

$$(176) \quad \begin{aligned} 0 < r_0^{min} &= \min\{r_0(t) : t \in \mathbb{S}^n\}; \quad 0 < r_2^{max} = \max\{r_2(t) : t \in \mathbb{S}^n\}; \\ 0 < u &= \min\{\phi_1(t) - \phi_2(t) : t \in \mathbb{S}^n\} < 1. \end{aligned}$$

By drawing a picture one sees that for big enough  $A > r_2^{max}$  we have

$$(177) \quad \forall \phi_2 \forall \phi_0 \quad \phi_2 < \phi_0 < \phi_2 + 1 \Rightarrow \arg_{(\phi_2, \phi_2+1)}(r_0^{min} \exp(i\pi\phi_0) + A \exp(i\pi\phi_2)) - \phi_2 < u.$$

This implication means that for any  $\delta > 0$  the set  $U'$ , given below, is contained in  $U$ :

$$(178) \quad U' = \left\{ (r_0^{min}, \delta, A, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{l} \phi_2 < \phi_1 < \phi_0 < \phi_2 + 1 \\ u \leq \phi_1 - \phi_2 \end{array} \right\} \subset U$$

where  $A, r_0^{min}, u$  are fixed in (176), (177) and we chose any  $\delta > 0$ . By (175) we see that for any  $t \in \mathbb{S}^n, s \in [0, 1]$  we have:

$$F(t, s) = (r_0(t)(1-s) + sr_0^{min}, r_1(t)(1-s) + s\delta, r_2(t)(1-s) + sA, \phi_0(t), \phi_1(t), \phi_2(t)) \in U.$$

Hence we obtain a continuous map  $F : \mathbb{S}^n \times [0, 1] \rightarrow U$ , which is a homotopy in  $U$  from the map  $\gamma$  to the following continuous map:

$$\gamma' : \mathbb{S}^n \rightarrow U \quad \gamma'(t) = (r_0^{min}, \delta, A, \phi_0(t), \phi_1(t), \phi_2(t)).$$

Furthermore, by (176) we have  $u \leq \phi_1(t) - \phi_2(t)$  for  $t \in \mathbb{S}^n$ , which means that  $\text{im}(\gamma') \subset U'$ . Since  $U'$  is contractible, there exists a homotopy in  $U'$  from  $\gamma'$  to a constant map. Since  $U' \subset U$  (see (178)), there exists a homotopy in  $U$  from  $\gamma$  to a constant map.

We show below that  $U$  is connected, and then by Whitehead theorem  $U$  is contractible.

Let  $x = (a_0, a_1, a_2, b_0, b_1, b_2) \in U$  and  $x' = (a'_0, a'_1, a'_2, b'_0, b'_1, b'_2) \in U$ . The formula (171) holds again, and by using it we can move continuously  $x'$  in  $U$  to a point  $x'' = (a'_0, a'_1, a'_2, b''_0, b''_1, b_2)$ . If we denote

$$0 < r_0^{min} = \min\{a_0, a'_0\}, 0 < r_2^{max} = \max\{a_2, a'_2\}, 0 < u = \min\{b_1 - b_2, b''_1 - b_2\} < 1$$

then choose  $A > r_2^{max}$  so that (177) holds with the chosen  $u, r_0^{min}, r_2^{max}$ , in particular for any  $\delta > 0$  the corresponding set  $U'$  defined by (178) is a subset of  $U$ . By the properties (175) and by the choice of  $u, r_0^{min}, A, \delta$  we can move the points  $x$  and  $x''$ , by changing only  $a_0, a_1, a_2, a'_0, a'_1, a'_2$ , continuously in  $U$  to points  $y, y'$  in  $U'$ , respectively. Now the connectivity of  $U$  follows from the connectivity of  $U'$ . The lemma is proved.  $\square$

**Corollary A.5.** *The set  $V$ , given below, is contractible:*

$$(179) \quad V = \left\{ (r_0, r_1, r_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{l} r_i > 0 \\ \phi_2 - 1 < \phi_1 < \phi_0 < \phi_2 \\ \arg_{(\phi_2-1, \phi_2)}(r_0 \exp(i\pi\phi_0) - r_2 \exp(i\pi\phi_2)) < \phi_1 \end{array} \right\}.$$

*Proof.* The assignment  $(a_0, a_1, a_2, b_0, b_1, b_2) \mapsto (a_0, a_1, a_2, b_0, b_1, b_2 - 1)$  maps homeomorphically the set  $V$  to the set  $U$  in Lemma A.4.  $\square$

**Remark A.6.** *If we have two contractible open subsets  $U, V$  in a f.d. manifold  $M$  and the intersection  $U \cap V$  is contractible, then by Seifert-van Kampen theorem, Mayer-Vietoris sequence, Hurewicz theorem and Whitehead theorem it follows that  $U \cup V$  is contractible.*

If  $U = \bigcup_{i \in A} U_i$  is an union of open subsets in a f.d. manifold  $M$  and for any finite subset  $F \subset A$  we have that  $\bigcup_{i \in F} U_i$  is contractible, then using Witehead theorem one can easily show that  $U$  is contractible as well.

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